Mathematics 351 – Abstract Algebra 1
Discussion 3 – Alternating groups; Cayley's Theorem
November 26, 2007

Background

Before the break, we saw that the symmetric group S_n contains a normal subgroup A_n whose elements are all of the "even" permutations, the permutations σ whose corresponding $n \times n$ permutation matrices $\rho(\sigma)$ have determinant 1:

$$A_n = \{\sigma \in S_n : \det(\rho(\sigma)) = 1\}.$$

A_n is called the *alternating group* on n letters. Today we will study the symmetric and alternating groups and develop some more of their properties.

Discussion Questions

I.

A) List all of the permutations in A_3 and A_4 using disjoint cycle decomposition notation.

B) Determine the group tables for A_3 and A_4. (*Comment:* The table for A_4 will be slightly tedious. You can "parallelize" the computation (split it up among your group members), though. This is an important group we have not seen before, so it is worth the effort!)

C) One fairly common *misconception* about Lagrange's Theorem is to think that if k divides $|G|$, then there must be a subgroup $H \subset G$ with $|H| = k$. But is that true for all of the divisors of $|A_4|$? Are there subgroups of A_4 of all the possible orders allowed by Lagrange's Theorem?

D) You should have seen that A_3 and A_4 have half the number of elements of S_3 and S_4 respectively. Prove in general that $|A_n| = \frac{1}{2}n!$.

II. One of the reasons that the symmetric groups are important examples is a famous result called *Cayley's Theorem*. In intuitive terms, this result says that if we understand the symmetric groups and their subgroups, then we understand *every finite group* G, because *every finite group* G *is isomorphic to a subgroup of* S_n *for some* n. In this problem, you will look at some examples and work through one proof of this theorem.

Let G be a finite group of order n and list the elements as

$$G = \{g_1, g_2, \ldots, g_n\} \quad (1)$$

in some order (the exact order is immaterial – any order will work as long as you maintain it consistently). As we know from a question on the last discussion, each row and column of the group table for G contains each element of G exactly once. If we look at the columns, this means that the *left multiplication mappings*

$$\begin{aligned} L_g : G &\to G \\ x &\mapsto gx \end{aligned}$$

are permutations of G (one-to-one, onto mappings from G to itself). We can identify each one with a permutation of $\{1, 2, \ldots, n\}$ via the listing of the group elements from (1) above.

A) Let G be a cyclic group of order 4, say with a as generator. Write down the permutation of G defined by L_g for each element of G, both in disjoint cycle notation, and as a permutation matrix.

B) Generalize your results from part A to cyclic groups of order n for any integer $n \geq 2$.

C) Repeat part A for $G = S_3$. (Be careful! What size will the permutation matrices be?)

Now we want to develop a proof of:

Cayley's Theorem. *Every finite group G is isomorphic to a subgroup of S_n for some n.*

The n here will actually be $n = |G|$. We introduce the following notation: Let $\widetilde{G}$ be the set of all the left multiplication mappings $L_g : G \to G$ by elements of G:

$$\widetilde{G} = \{L_g : g \in G\},$$

viewed as permutations in S_n.

D) Show that $\widetilde{G}$ is a group under the operation of composition of functions. Hence it can be viewed as a subgroup of S_n by identifying each L_g mapping with its corresponding permutation in S_n.

E) Show that the mapping $\varphi : G \to \widetilde{G}$ defined by $\varphi(g) = L_g$ is an *isomorphism of groups.* This completes the proof of Cayley's Theorem.