Mathematics 352 – Abstract Algebra II
Solutions for Problem Set 2
January 28, 2008

Section 8.2

22. We prove the result first for finite abelian p-groups for an arbitary prime p. Let G and H be two finite abelian p-groups such that for all $m \geq 1$, the numbers of elements of order m in G and H are the same. The m's involved in this case are powers of p, of course. Since G and H are finite, they have the same number of elements of the same maximal order p^M for some M. then writing

$$N_i(G) = |\{g \in G : |g| = p^i\}|,$$

(the number of elements of order p^i), we have

$$|G| = \sum_{i=0}^{M} N_i(G) = \sum_{i=0}^{M} N_i(H) = |H|.$$

We will argue *by induction* on the order of the groups that G and H must be isomorphic under the given hypothesis. If $|G| = |H| = 1$, then there is nothing to prove since $G = \{0_G\}$ and $\{0_H\} = H$ are both trivial groups hence are isomorphic.

Now assume the result has been proved for p-groups of all orders $< p^k$ and consider G, H of order p^k such that for all $m \leq p^k$, the numbers of elements of order m in G and H are the same. By hypothesis, the maximal orders are the same, say p^M $(M \leq k)$, and there are the same numbers of elements in G and H of order p^M. Pick one such element $a \in G$ and one such element $b \in H$. By Lemma 8.6 (used in the proof of the Structure Theorem), we have $G = \langle a \rangle \oplus K$ and $H = \langle b \rangle \oplus L$ for some subgroups K, L. Now, K, L are p-groups of order p^{k-M}.

Moreover, we claim that they must also satisfy the hypothesis – K and L must contain the same number of elements of order p^i for each p^i. We will prove this by a separate induction argument. The base case is $i = 0$ and the statement is true in that case since there is only one element of order 1 in each group (the 0 elements). Suppose we know that

$$N_j(K) = N_j(L), \quad \text{for all } 0 \leq j \leq i-1.$$

Consider the numbers of elements of order p^i in K, L, G, and H. By the direct sum decompositions for G, any element of order p^i in G must look like $b+c$ where $b \in \langle a \rangle$ and $c \in K$ and $\text{lcm}(|b|, |c|) = p^i$ (that is, at least one of $|b|, |c|$ is p^i and the other is $\leq p^i$). We count how many elements this gives, writing $C = \langle a \rangle$:

$$N_i(G) = (N_0(C) + \cdots + N_i(C))N_i(K) + (N_0(K) + \cdots + N_{i-1}(K))N_i(C).$$

Similarly from the direct sum decomposition $H = \langle b \rangle \oplus L = D \oplus L$ we get

$$N_i(H) = (N_0(D) + \cdots + N_i(D))N_i(L) + (N_0(L) + \cdots + N_{i-1}(L))N_i(C).$$

Now C and D are isomorphic since they are cyclic groups of the same order. This means $N_j(C) = N_j(D)$ for all j. The induction hypothesis shows that the terms in the second sets of parentheses are equal. Moreover $N_i(G) = N_i(H)$ by hypothesis. Therefore $N_i(K) = N_i(L)$ and this finishes the inner induction argument. This shows that K and L satisfy the same hypothesis as G and H.

Hence by the induction hypothesis (for the outer induction), $K \simeq L$. Since $C \simeq D$ as above, this shows that $G \simeq H$.

Now, consider the general case, where G, H are finite abelian groups, but not necessarily p-groups. The same reasoning used for p-groups (but applied to elements of all possible orders) shows that $|G| = |H|$. The hypothesis implies in particular that for any prime p, the number of elements of order p^i for each i is the same in both G and H. Letting i range over all integers $i \geq 0$, these elements are exactly the elements of the subgroups $G(p)$ and $H(p)$. The above proof shows that $G(p) \simeq H(p)$. This is true for all p dividing the order of G, H. Hence by the Structure Theorem $G \simeq H$.

Section 8.3

2. (a) Since $|S_4| = 24 = 2^3 \cdot 3$, Sylow 2-subgroups are subgroups of order $2^3 = 8$. We can get one such subgroup by taking the cyclic subgroup generated by the 4-cycle (1234) and including additional suitable elements of order 2 to make subgroups isomorphic to D_4:

$$H_1 = \{(), (1234), (13)(24), (1432), (24), (13), (12)(34), (14)(23)\}$$

Then, other subgroups of the same order can be generated by conjugating this one. For instance conjugating by (12) we get

$$H_2 = \{(), (1342), (23)(14), (1243), (14), (23), (12)(34), (24)(13)\}.$$

Conjugating by (23) we get

$$H_3 = \{(), (1324), (12)(34), (1423), (34), (12), (13)(24), (34)(12)\}.$$

(b) The Sylow 3-subgroups have order 3; they are cyclic, generated by the 3-cycles:

$$\begin{aligned} K_1 &= \{(), (123), (132)\} \\ K_2 &= \{(), (124), (142)\} \\ K_3 &= \{(), (134), (143)\} \\ K_4 &= \{(), (234), (243)\}. \end{aligned}$$

3. Since $|A_4| = 12 = 2^2 \cdot 3$, the Sylow 2-subgroups have order 4. There is exactly one such subgroup:

$$H = \{(), (12)(34), (13)(24), (14)(23)\}.$$

The Sylow 3-subgroups have order 3; they are cyclic, generated by the 3-cycles (this is the same as in S_4):

$$\begin{aligned} K_1 &= \{(), (123), (132)\} \\ K_2 &= \{(), (124), (142)\} \\ K_3 &= \{(), (134), (143)\} \\ K_4 &= \{(), (234), (243)\}. \end{aligned}$$

Comment: Note by Sylow 3 that the number of 2-subgroups must be an odd number dividing 12. The possible numbers are 1 and 3, but there are only 3 elements of order a power of 2 in A_4.

4. The elementary divisors are $2, 2^2, 2^2, 3, 3, 5$. There is exactly one Sylow 2-subgroup, isomorphic to $Z_2 \oplus Z_4 \oplus \mathbb{Z}_4$. There is one Sylow 3-subgroup, isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. There is one Sylow 5-subgroup, isomorphic to $\mathbb{Z}_5$.

5. (a) By Sylow theorem 3, the number of 3-subgroups is congruent to 1 mod 3 and divides 72. The number could be 1 or 4. (b) Similarly, the number is congruent to 1 mod 5 and divides 60. Hence the possibilities are 1 or 6.

6. (b) $143 = 11 \cdot 13$. By Sylow 3, the number of Sylow 11-subgroups is 1. Similary, the number of 13-subgroups is 1. By Sylow theorem 2, both of these subgroups are normal. Hence $G \simeq H \times K$ is cyclic of order 143.

Section 8.4

1 (a) Use the permutation form of D_4 from 8.3/2a above:

$$D_4 \simeq \{(), (1234), (13)(24), (1432), (24), (13), (12)(34), (14)(23)\} \subset S_4$$

to make computing conjugates easier(!) The conjugacy classes are

$$\begin{aligned} C_1 &= C_{()} = \{()\} \quad \text{in the center} \\ C_2 &= C_{(1234)} = \{(1234), (1432)\} \\ C_3 &= C_{(13)(24)} = \{(13)(24)\} \quad \text{also in the center} \\ C_4 &= C_{(12)(34)} = \{(12)(34), (14)(23)\} \\ C_5 &= C_{(13)} = \{(13), (24)\}. \end{aligned}$$

There is no real "trick" to this – just compute!

(c) The conjugacy classes in

$$A_4 = \{(), (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$$

are

$$\begin{aligned} C_1 &= \{()\} \\ C_2 &= \{(12)(34),(13)(24),(14)(23)\} \\ C_3 &= \{(123),(243),(134),(142)\} \\ C_4 &= \{(132),(234),(143),(124)\}. \end{aligned}$$

(Note in particular that the two elements of order 3 in each Sylow 3-subgroup end up in different conjugacy classes in A_4. This is true because the permutations that conjugate (123) to (132) in S_4, like (23), are *odd* – they are not in A_4. Hence the conjugacy classes in S_4 split up into smaller subsets in A_4.)