Mathematics 352 – Abstract Algebra 2
Information on Final Exam
April 24, 2008

General Information, Format and Groundrules

The final exam for this course will be given during the regular scheduled period for 11:00 MWF classes – 8:30am - 11:30am on Saturday, May 3. This will be an *closed book* exam. The exam will be written so that if you are well prepared and work steadily, you should take about 2 hours to complete it, but you will have the full 3-hour exam period to work on the exam, so this should eliminate most of the element of time pressure.

Topics

The final exam will be a *comprehensive exam.* It will cover all of the material we have discussed since the beginning of the semester, giving roughly equal attention to the sections of the class corresponding to the two midterms and also the definition of solvable groups and the question of which polynomial equations are solvable by radicals.

Like the midterm exams, this one will focus mainly on the key definitions and theorems we have studied. There will also be a few problems or parts of problems where you will need to apply some of those definitions and theorems to new situations.

The topics to be covered are:

1. The structure theorem for finite abelian groups: Every finite abelian G is isomorphic to a direct sum of cyclic groups of prime power order. Know the classifications of these via the *elementary divisors* and the *invariant factors* and how to go back and forth between both forms.

2. The Sylow theorems and the background needed for their proof – the conjugacy relation on a group, the class equation of G and the role of the center $Z(G)$, centralizers of $a \in G$, the formula $|C_a| = [G : C(a)] = |G|/|C(a)|$ (that is: the order of the conjugacy class of a is equal to the index of the centralizer of a), normalizers of subgroups, and so forth.

3. Applications of the Sylow theorems to structure of finite groups, dihedral groups, etc.

4. Vector spaces over a field, including examples where the vector space is an extension field.

5. Extension fields, simple extensions (that is $F(u)$ for u), algebraic and transcendental elements over F, minimal polynomials of algebraic elements, how to find $[F(u) : F]$ when u is algebraic over F.

6. Algebraic extensions and their properties.

7. Splitting fields of $f(x) \in F[x]$ and normal extensions.

8. Separable polynomials and extensions, the Primitive Element Theorem (Theorem 10.18).

9. Finite fields, the prime field, constructing fields of order p^n for all $n \geq 1$. All fields of order p^n are isomorphic. A field of order p^n has subfields of order p^k if and only if $k \mid n$.
10. Geometric constructions, constructible numbers, the characterization of constructible numbers: α is constructible if and only if $[\mathbf{Q}(\alpha) : \mathbf{Q}] = 2^m$ for some m. The three Greek construction problems and their resolution.
11. F-automorphisms of an extension field K of F, the Galois group $\mathrm{Gal}_F K$,
12. The Galois correspondence between subgroups of $\mathrm{Gal}_F K$ and intermediate fields E with $F \subseteq E \subseteq K$, Galois (= finite-dimensional, separable, and normal) extensions, the Fundamental Theorem of Galois Theory.
13. Root towers, solvable groups, the Galois Criterion for solvability by radicals. Non-solvability of S_5 and A_5, and applications to quintic polynomials. *Note:* Problems 11 - 18 in Section 11.3 are good here!

Review Session

I would be happy to schedule a day or evening review session before the exam, but I have an off-campus commitment in the evening on Thursday, May 1, so that evening will not work.

Key Theorems and Proofs to Know

1. Know the statement of the structure theorem for finite abelian groups (Theorem 8.7 in Hungerford).
2. Know the statements of the three Sylow theorems and the proof of Sylow 1: If p is a prime and $p^n \| G|$, then G contains a subgroup of order p^n.
3. Know the proof of the multiplicativity of degrees in towers of field extensions: If $F \subseteq E \subseteq K$ are field extensions, and $[E : F]$ and $[K : E]$ are finite, then $[K : F] = [K : E][E : F]$ (Theorem 10.4 in Hungerford).
4. Know the proof of the main theorem on simple extensions (Theorem 10.7 in Hungerford): $F(u) \simeq F[x]/(p(x))$ where $p(x)$ is the minimal polynomial of u over F, and $[F(u) : F] = n$ if $\deg p(x) = n$.
5. Know the proof that $[K : F]$ finite implies every element of K is algebraic over F (Theorem 10.9 in Hungerford).
6. Know how to show that the splitting field of $x^{p^n} - x$ over $F = \mathbf{Z}_p$ is a field of order p^n. (This is one implication from Theorem 10.25.)
7. Know the statement of the Fundamental Theorem of Galois Theory (Theorem 11.11 in Hungerford). Also know the statement and proof of the Lemma used in the proof of the Fundamental Theorem (Lemma 11.7 in Hungerford).

8. Know the statements and the proofs of the two theorems from class describing the structure of Galois groups of radical extensions. Let p be prime in both of the following:

   (a) Let F be a field containing a primitive pth root of 1, ζ_p. If $K = F(u)$ where $u^p = a$ for some $a \in F$ that is not a pth power in F, then $\text{Gal}_F K$ is a cyclic group of order p (isomorphic to the additive group $(\mathbb{Z}_p, +)$).

   (b) If ζ_p is a primitive pth root of 1, then $\text{Gal}_{\mathbb{Q}}\mathbb{Q}(\zeta_p)$ is a cyclic group of order $p-1$ (isomorphic to the multiplicative group $(\mathbb{Z}_p^*, \cdot)$).

Philosophical Comments and Suggestions on How to Prepare

- The reason we give final exams in almost all mathematics classes is to give students a final opportunity to "put whole courses together" in their minds. This course is an especially central one connecting many key ideas in mathematics and illustrating their historical development. And in fact, the ideas we have learned lead directly to many areas of current research and recent "big theorems" in the subject.

- There is a lot here, but there are also many, many connections between different portions of what we have done – identifying them and understanding how they fit together is a good thing to do at some point. (One small example: How does the Structure Theorem for finite abelian groups show that for any finite abelian group G, there exists a chain of subgroups

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset \{e\}$$

  where G_i/G_{i+1} is cyclic of prime order for each i, hence that G satisfies one form of the definition of solvability?)

- Reread your class notes in addition to the text, especially for topics where you lost points on the midterms. There are a lot of worked-out examples and discussions of all of the topics we have covered there.

- Get started reviewing early and do some work on this *every day* between now and the date of the final. Don't try to "cram" everything in at the end.

- Look over the midterm exams with the solutions. Go over your corrected problem sets. If there were questions where you lost a lot of points, be sure you understand why what you did was not correct, and how to solve those questions.

- Be sure you actually do enough practice problems so that you have the facility to solve exam-type questions in a limited amount of time. *Even if you have saved solutions for practice problems from the midterms*, it is going to be much more beneficial to do practice problems starting "from scratch" rather than just reading old solutions.