Mathematics 352 – Abstract Algebra II
Discussion 2 – More on Galois Groups
March 28, 2008

Background

Given an extension field K of a field F, recall that we have defined

$$\text{Gal}_F K = \{\sigma : K \to K : \sigma \text{ is an } F\text{-automorphism of } K.\ \}$$

which is a group under the operation of function composition, called the *Galois group* of K over F. For instance, last Friday we saw that $\text{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \{id, \rho, \sigma, \tau\}$ is a noncyclic group of order 4, isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$. Today we want to work out another example and start to understand a relationship between subgroups of $\text{Gal}_F K$ and subfields of K containing F.

Discussion Questions

A) Consider the polynomial $f(x) = x^3 - 2 \in \mathbb{Q}[x]$, and let K be the splitting field of $f(x)$ over $\mathbb{Q}$.

(1) Show that $K = \mathbb{Q}(\sqrt[3]{2}, \omega)$, where $\omega = \dfrac{-1 + i\sqrt{3}}{2}$ is a cube root of 1.

(2) Consider the minimal polynomials $f(x) = x^3 - 2$ of $\sqrt[3]{2}$ and $g(x) = x^2 + x + 1$ of ω in $\mathbb{Q}[x]$. By the theorem we finished proving in class last Friday, we know that given any root v of $f(x)$ and any root w of $g(x)$, there must be some element ϕ of $\text{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt[3]{2}, \omega)$ with $\phi(\sqrt[3]{2}) = v$ and $\phi(\omega) = w$. How many distinct elements of the $\text{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt[3]{2}, \omega)$ are there? List them all.

(3) By numbering the 3 roots of $x^3 - 2$ in some way, establish a 1-1 correspondence between $\text{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt[3]{2}, \omega)$ and a subgroup of S_3. (Which permutation of the roots is induced by each element of the Galois group?) What is the structure of $\text{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt[3]{2}, \omega)$? (i.e. to which "standard" group is it isomorphic?)

B) Now we want to consider subgroups of $\text{Gal}_F K$ and subfields of K.

(1) Show in general that if $G = \text{Gal}_F K$ and H is a subgroup of G, then

$$E_H = \{a \in K : \sigma(a) = a \text{ for all } \sigma \in H\}$$

is a *subfield* of K containing F (called the *fixed field* of H for a hopefully clear reason!). Note that this requires you to show that E_H is a subring of K which is also a field.

(2) Let $F = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. The Galois group $G = \text{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt{2}, \sqrt{3})$ has 5 different subgroups H (including the "trivial" subgroups $\{id\}$ and G). What are the fixed fields of each of them?

(3) The Galois group $G = \text{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt[3]{2}, \omega)$ has 6 different subgroups H (again including the trivial subgroups). Find the fixed fields of each one.

Assignment

Group writeups due in class on Monday, April 7.