Mathematics 352 – Abstract Algebra II
Discussion 1 – Vector Spaces
Friday, February 8, 2008

Background

Recall from Linear Algebra that a vector space over a field F is a set V of "vectors" with a sum operation and a multiplication operation by scalars in F. Today we want to review some of the properties of vector spaces and see how they apply to some new examples related to the topics we have been studying in this course.

Discussion Questions

A) General properties.

1) What are the properties of the sum and scalar multiplication operations that define the vector space structure? Try to express these in as economical form as possible (this might look different from the form you saw in the Linear Algebra course).
2) What is a *basis* for a vector space V?
3) It is a very important general theorem about vector spaces that every vector space over a field F has a basis, and that all bases have the same cardinality. One nice proof of this fact in the case that the vector space V has some finite spanning set is based on the following lemma:

   Lemma. Let V be a vector space that has a finite spanning set S. If T is any linearly independent subset of V, then $|T| \le |S|$.

   Prove the lemma and use it to deduce that if B_1 and B_2 are both finite sets that are bases for a vector space V, then $|B_1| = |B_2|$. Where exactly are you using that the scalars come from a field?
4) What important invariant of vector spaces is defined on the basis of what you showed in part 3?

The field K that is part of a vector space structure can be any field – in particular, this means that examples such as $K = \mathbf{Z}_p$ or even $K = F[x]/\langle p(x)\rangle$ where $p(x)$ is an irreducible polynomial in $F[x]$ for a field like $F = \mathbf{Z}_p$ are possibilities! Moreover, the elements of a vector space do not necessarily look like "vectors" in the usual sense of an ordered n-tuple of elements of the field.

B) Some new examples of vector spaces.

1) Show that the set of all polynomials in two variables x, y with coefficients in $F = \mathbf{Z}_3$, of total degree ≤ 3 in x and y forms a vector space over the field F. What is the dimension of this vector space?
2) Let $K = \mathbf{Z}_2[x]/\langle x^3 + x + 1\rangle$ Show that K forms a vector space over the field $\mathbf{Z}_2$. What is the dimension of this vector space?

Note: $p(x) = x^3 + x + 1$ is irreducible in $\mathbf{Z}_2[x]$. This implies, by a theorem from last semester, that V is a field. *A field K can be a vector space over another field F (!)* We will be using this observation extensively! Another similar example,

3) Let

$$V = \mathbf{Q}[\sqrt{3}] = \{a + b\sqrt{3} : a, b \in \mathbf{Q}\}.$$

Show that V is a vector space over $F = \mathbf{Q}$ and determine its dimension.

Assignment

Writeups due in class on Wednesday, February 13.