Mathematics 243, section 3 – Algebraic Structures
Problem Set 9
due: Friday, November 30

'A' Section

1. For each of the following values of n,
   - Find all distinct generators of the group $(\mathbb{Z}_n, +)$,
   - Find all subgroups of $(\mathbb{Z}_n, +)$ and their orders
   - Find all elements of $(\mathbb{Z}_n^\times, \cdot)$ and their orders (for the *multiplication operation* mod n now)

   $$n = 13, 16, 30$$

   (Use the "big theorem" on cyclic groups for as much of this as possible. It is not necessary to do a lot of computations in most cases.)

2. Let $\varphi : \mathbb{Z}_{18} \to \mathbb{Z}_9$ be defined by $\varphi([x]) = [3x]$.
   (a) Verify that φ is a group homomorphism.
   (b) Determine the kernel of φ.
   (c) Determine the image of φ.

'B' Section

1. Let G be a group and consider the mapping $\varphi : G \to G$ defined by $\varphi(x) = x^{-1}$. Show that φ is always one-to-one and onto, but that φ is an isomorphism of groups if and only if G is an *abelian* group.

2. An *automorphism* of a group G is an isomorphism of groups $\varphi : G \to G$ (that is, the domain and the range are both the same group G).
   (a) Let $A = \{a, b, c\}$ and $G = \mathcal{S}(A)$ be the group of permutations of A. Show that $\varphi : G \to G$ defined by $\varphi(f) = R_a \circ f \circ R_a$ is an automorphism of G.
   (b) Show that the collection of all automorphisms of a general group G is itself a group under the operation of function composition.
   (c) Show if G is a general group and $g \in G$, then the conjugation mapping defined by $\varphi_g(x) = gxg^{-1}$ is an automorphism of G. (Note that the example in part (a) has this form.)
   (d) Show that the collection of φ_g for all $g \in G$ (as in part (c)) is a *subgroup* of the group of automorphisms of G.

3. We can consider isomorphism of groups as a relation on the collection of all groups: $GRH \Leftrightarrow$ there exists an isomorphism $\varphi : G \to H$. Show that isomorphism of groups is an *equivalence relation* on the collection of all groups.

4. Let $G = \langle a \rangle$ be a cyclic group and let $\varphi : G \to H$ be a group homomorphism. Show that if we know the one element $\varphi(a)$, then we know where φ maps every element of G.