Mathematics 243, section 3 – Algebraic Structures
Problem Set 8
due: Friday, November 16

'A' Section

1. Consider the 2×2 matrices I_2, S, X, Y, D, T defined by

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Let $G = \{I_2, S, S^2, S^3, X, Y, D, T\}$. Construct the operation table for matrix multiplication on this set and verify that G is a group under this operation.

2. a. Let $G = \mathbb{Z}_{12}$ under the operation of *addition* modulo 12. Determine the cyclic subgroups $\langle [a] \rangle$ for all $[a] \in \mathbb{Z}_{12}$.

   b. Conjecture a general formula for the number of elements in $\langle [a] \rangle$ in terms of the integers a and 12. Check out your conjecture on the corresponding list of cyclic subgroups of $\mathbb{Z}_{20}$ constructed in class on November 14.

'B' Section

1. Let

$$G = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

Show that G is a group under the operation of matrix addition.

2. Let

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

Is G is a group under the operation of matrix multiplication. If so, say why; if not say which properties in the definition fail.

3. Let G be a group, and consider the relation R on G defined by $xRy \leftrightarrow y = axa^{-1}$ for some $a \in G$. (If xRy is true, the a that works in the equation will depend on x, y.) R is called the *conjugacy* relation on G.

a. Show that conjugacy is an equivalence relation on G.

b. Show that G is an abelian group if and only if the equivalence classes for the conjugacy relation satisfy $[x] = \{x\}$ for all $x \in G$.

c. More generally, show that if $[x] = \{x\}$ for some element $x \in G$, then $xa = ax$ for all $a \in G$ and conversely. The set of such elements x is called the *center* of G: The center is the subset of G defined by:

$$Z(G) = \{x \in G \mid xa = ax \text{ for all a in } G\}.$$

d. Show that the center of G (as defined in part c) is a subgroup of G.

e. Let $x \in G$ be a fixed element and define $C_x = \{a \in G \mid x = axa^{-1}\}$. Show that C_x is a subgroup of G (C_x is called the *centralizer of* x).

f. Let $G = \mathcal{S}(A)$ be the group of permutations of $A = \{a, b, c\}$. Using the names for the elements of this group we introduced in Problem Set 3, find all of the equivalence classes for the conjugacy relation on G (there are three of them), determine the centralizer of each element of G, and the center of G. How are the sizes of the equivalence class of x and the number of elements of the centralizer of x related in each case?

4. Let H and K be subgroups of a group G.

a. Show that $H \cap K$ is also subgroup of G.

b. Find an example where $H \cup K$ *is* a subgroup of G and one where $H \cup K$ *is not* a subgroup of G.