1. Let $A = \{a, b, c\}$. In class we discussed the set $S(A)$ of permutations of $A$, and we wrote

$$S(A) = \{I_A, R_a, R_b, R_c, C_1, C_2\} \quad (1)$$

where the $R$’s were mappings that fixed one element and swapped the other two, and the $C$’s were the cyclic permutations. For instance $R_a(a) = a, R_a(b) = c$, and $R_a(c) = b$ and $C_1(a) = b, C_1(b) = c$ and $C_1(c) = a$. We know that $\circ$ defines a binary operation on $S(A)$ by results from class on September 14 and 17. Make the operation table for $\circ$ on this set, showing the results of composing all pairs of elements of $S(A)$, listing the labels for the rows and columns in the order given in Eq. (1) above. The entry in the row with label $f$ and the column with label $g$ should be the mapping $f \circ g$ in all cases.

2. Let

$$A = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & -3 & -4 \\ 1 & -8 & 2 \\ 0 & 1 & 4 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 3 & 3 \\ 5 & 0 & -2 \end{pmatrix}.$$

Perform the indicated matrix operations, if possible. If the operation is not possible, say why not.

(a) $A + D$
(b) $AC$
(c) $AD + A$
(d) $CB + AB$

3. Let $S$ be the following set of matrices in $M_{3\times3}(\mathbb{R})$:

$$S = \{I_3, F_x, F_y, F_z, G_1, G_2\}, \quad (2)$$

where $I_3$ is the $3 \times 3$ identity matrix,

$$F_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad F_z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

and

$$G_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$
Show that matrix multiplication defines a binary operation on this set $S$ by constructing an operation table showing the results of doing all possible products of pairs of elements in $S$. Label the rows and columns in the order given in Eq. (2) above. Do you notice something about this table and the one in question 1?

'B' Section

1. Let $A$ be a set and $f, g \in S(A)$ be permutations.
   (a) Show that $(f^{-1})^{-1} = f$ as mappings on $A$.
   (b) Show that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ as mappings on $A$.

2. Suppose that $A$ is a finite set. Show:
   (a) A mapping $f : A \to A$ is a permutation of $A$ if and only if $f$ is one-to-one.
   (b) A mapping $f : A \to A$ is a permutation of $A$ if and only if $f$ is onto.

   Hint for both parts: The idea here is to show that each of the defining properties of a permutation implies the other in this special situation. The ideas used in the proof of Exercise B 2 on Problem Set 1 may be helpful.

3. Show that matrix addition and matrix multiplication both define binary operations on the set of matrices

   $M = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$.

   (That is, show that if $A, B$ are two general matrices in $M$, then $A + B \in M$ and $A \cdot B \in M$.)

4. Let

   $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

   be any matrix in $M_{2\times2}(\mathbb{R})$ such that $\Delta = ad - bc \neq 0$. What happens if you multiply $AB$ and $BA$ for the matrix

   $B = \begin{pmatrix} d & -b \\ -c \Delta & \Delta \end{pmatrix}$?

   What does this say about $B$?