Mathematics 243, section 3 – Algebraic Structures
Problem Set 3
due: September 21, 2012

'A' Section

1. Let $\mathcal{A} = \{a, b, c\}$. In class we discussed the set $\mathcal{S}(\mathcal{A})$ of permutations of $\mathcal{A}$, and we wrote

$$\mathcal{S}(\mathcal{A}) = \{I_{\mathcal{A}}, R_a, R_b, R_c, C_1, C_2\} \tag{1}$$

where the R's were mappings that fixed one element and swapped the other two, and the C's were the cyclic permutations. For instance $R_a(a) = a$, $R_a(b) = c$, and $R_a(c) = b$ and $C_1(a) = b$, $C_1(b) = c$ and $C_1(c) = a$. We know that $\circ$ defines a binary operation on $\mathcal{S}(\mathcal{A})$ by results from class on September 14 and 17. Make the operation table for $\circ$ on this set, showing the results of composing all pairs of elements of $\mathcal{S}(\mathcal{A})$, listing the labels for the rows and columns in the order given in Eq. (1) above. The entry in the row with label f and the column with label g should be the mapping $f \circ g$ in all cases.

2. Let

$$A = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 3 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}, \ C = \begin{pmatrix} 3 & -3 & -4 \\ 1 & -8 & 2 \\ 0 & 1 & 4 \end{pmatrix}, \ D = \begin{pmatrix} 4 & 3 & 3 \\ 5 & 0 & -2 \end{pmatrix}.$$

Perform the indicated matrix operations, if possible. If the operation is not possible, say why not.

(a) $A + D$
(b) AC
(c) $AD + A$
(d) $CB + AB$

3. Let S be the following set of matrices in $M_{3\times 3}(\mathbb{R})$:

$$S = \{I_3, F_x, F_y, F_z, G_1, G_2\}, \tag{2}$$

where I_3 is the 3×3 identity matrix,

$$F_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ F_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ F_z = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$G_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ G_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Show that *matrix multiplication* defines a binary operation on this set S by constructing an operation table showing the results of doing all possible products of pairs of elements in S. Label the rows and columns in the order given in Eq. (2) above. Do you notice something about this table and the one in question 1?

'B' Section

1. Let $\mathcal{A}$ be a set and $f, g \in S(\mathcal{A})$ be permutations.
   (a) Show that $(f^{-1})^{-1} = f$ as mappings on $\mathcal{A}$.
   (b) Show that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ as mappings on $\mathcal{A}$.
2. Suppose that $\mathcal{A}$ is a *finite* set. Show:
   (a) A mapping $f : \mathcal{A} \to \mathcal{A}$ is a permutation of $\mathcal{A}$ if and only if f is one-to-one.
   (b) A mapping $f : \mathcal{A} \to \mathcal{A}$ is a permutation of $\mathcal{A}$ if and only if f is onto.
   Hint for both parts: The idea here is to show that each of the defining properties of a permutation *implies the other* in this special situation. The ideas used in the proof of Exercise B 2 on Problem Set 1 may be helpful.
3. Show that matrix addition and matrix multiplication both define binary operations on the set of matrices
$$M = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}.$$
   (That is, show that if A, B are two general matrices in M, then $A + B \in M$ and $A \cdot B \in M$.)
4. Let
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
   be any matrix in $M_{2\times 2}(\mathbb{R})$ such that $\Delta = ad - bc \neq 0$. What happens if you multiply AB and BA for the matrix
$$B = \begin{pmatrix} \frac{d}{\Delta} & -\frac{b}{\Delta} \\ -\frac{c}{\Delta} & \frac{a}{\Delta} \end{pmatrix}?$$
   What does this say about B?