Mathematics 243, section 3 – Algebraic Structures
Problem Set 2
due: September 14, 2012

'A' Section

1. Let $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be the indicated functions. In each case, say whether $f, g, f \circ g, g \circ f$ are one-to-one (injective) or onto (surjective), both, or neither. Justify your answers.

   a. $f(x) = 3x$, $g(x) = 4 - x$

   b. $f(x) = |x|$, $g(x) = \begin{cases} x & \text{if } x \text{ is even} \\ x - 1 & \text{if } x \text{ is odd} \end{cases}$.

2. Consider the binary operation on $\mathbb{Z}$ given by

$$x * y = x + y + 5$$

   a. Is $*$ commutative? Why or why not?

   b. Is $*$ associative? Why or why not?

   c. Is there an identity element for $*$. What is the identity, or why not?

   d. Are there any elements of $\mathbb{Z}$ that have inverses under this operation? What are they, and what are the inverses?

3. Let $A = \{x, y, z, w\}$ and let $*$ be the binary operation on A given by the following table:

$*$	x	y	z	w
x	x	y	z	w
y	y	y	w	w
z	z	w	z	w
w	w	w	w	w

   a. Explain how you can tell this operation is commutative.

   b. Explain why x is an identity element for $*$.

   c. Which elements have inverses and what are the inverses?

   d. What is $(y * z) * z$? Is that the same as $y * (z * z)$?

'B' Section

1. Let $f : A \to B$ and $g : B \to A$ be mappings. Prove that if $f \circ g$ is onto and $g \circ f$ is one-to-one, then f is one-to-one *and* onto.

2. Let $*$ be an associative binary operation on a set A and assume there is an identity element e for $*$. If $a \in A$ has inverses b_1 and b_2, show that $b_1 = b_2$. Hint: Consider the "product" $(b_1 * a) * b_2$.

3. Let A be a set and let $\mathcal{P}(A)$ be the power set of A as defined in §1 of the text and on Problem Set 1. Let $*$ be the binary operation on $\mathcal{P}(A)$ defined by $S * T = S \cup T$. Answer the following questions and prove your assertions.

   a. Is $*$ associative? Is $*$ commutative?

   b. Is there an identity element in $\mathcal{P}(A)$ for this operation?

   c. What elements of $\mathcal{P}(A)$ have inverses for this operation?

   d. Make a table like the one in problem 3 of the 'A' section for the operation in this problem, when $A = \{a, b\}$. List the elements of $\mathcal{P}(A)$ in this order on the borders of the table:

   $$\emptyset, \{a\}, \{b\}, \{a, b\}.$$

   Do you notice something?

4. Let $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ be the vector *cross product* from multivariable calculus (MATH 241). Recall that this operation is defined by the following formula:

   $$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2 b_3 - a_3 b_2, -(a_1 b_3 - a_3 b_1), a_1 b_2 - a_2 b_1).$$

   a. Show that $\times$ is not associative and not commutative.

   b. Show that $\times$ does satisfy the *Jacobi identity*:

   $$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a})$$

   for all $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ in $\mathbb{R}^3$. The $+$ in this formula means the vector sum in $\mathbb{R}^3$, defined for vectors $\mathbf{d} = (d_1, d_2, d_3)$ and $\mathbf{e} = (e_1, e_2, e_3)$ by the rule

   $$\mathbf{d} + \mathbf{e} = (d_1, d_2, d_3) + (e_1, e_2, e_3) = (d_1 + e_1, d_2 + e_2, d_3 + e_3).$$

   c. *Extra Credit* In a sense, the additional term $\mathbf{b} \times (\mathbf{c} \times \mathbf{a})$ on the right in the Jacobi identity measures the failure of associativity. Using that idea, is $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ *ever* equal to $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ when all three of the vectors are nonzero? Explain. Hint: One way to approach this is to think about the geometric conditions on the three vectors under which it will be true that

   $$\mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = \mathbf{0}.$$