Mathematics 243, section 3 – Algebraic Structures
Problem Set 1
due: September 7, 2012

'A' Section

1. Let

$$U = \{1, 2, 3, \ldots, 12\}$$
$$A = \{1, 3, 5, 7, 9, 11\}$$
$$B = \{4, 6, 8\}$$
$$C = \{7, 8, 9, 10\}$$

Find each of the following sets:

a. $A' \cap C$
b. $A \cup C'$
c. $A \cap (B \cup C)$
d. $(A - B) \cup (B - C)$

2. Let $S = \{a, b, c, d\}$

a. List all elements of $\mathcal{P}(S)$ (the power set)
b. Write out all of the *partitions* of S.

3. Let $A = \{1, 2\}$. For each pair of subsets $S, T \subset A$ (including the cases where S or T is $\emptyset$ or A itself), define $S + T = (S \cup T) - (S \cap T)$. Make a chart with rows labeled with the possibilities for S, columns labeled with the possibilities for T, and entries showing which subset of A is produced as $S + T$. (For example, in the row for $S = \{1\}$ and the column for $T = \{1, 2\}$, your chart will contain the entry $S + T = \{1, 2\} - \{1\} = \{2\}$.)

4. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c, d, e\}$ and let $f : A \to B$ be defined by $f(1) = a$, $f(2) = e$, $f(3) = c$, $f(4) = a$, and $f(5) = e$. Find the following:

a. $f(A)$
b. $f(S)$ for $S = \{3, 4, 5\} \subset A$
c. $f^{-1}(f(S))$ for $S = \{1, 2, 3\} \subset A$
d. $f^{-1}(T)$ for $T = \{c, d, e\} \subset B$
e. $f(f^{-1}(T))$ for $T = \{a, b, c\} \subset B$

5. Let P be the set of positive integers and let $f : P \to P$ be defined by

$$f(x) = \begin{cases} 3x + 1 & \text{if } x \text{ is odd} \\ x/2 & \text{if } x \text{ is even} \end{cases}$$

a. Compute $f(7)$, $f(f(7))$, $f(f(f(7)))$, etc. What happens eventually if you continue this long enough? Try the same thing with $f(9)$, $f(f(9))$, $f(f(f(9)))$, etc.

b. Show by example that f is *not* one-to-one.

c. Explain why f *is* onto.

'B' Section

1. Let A, B be any two subsets of a universal set U.

   a. Prove that $(A \cap B)' = A' \cup B'$.

   b. Prove that $(A \cup B) - (A \cap B) = (A \cap B') \cup (A' \cap B)$.

2. Let A and B be finite sets with $|A| = m$ and $|B| = n$.

   a. Show that if $m > n$, there are no one-to-one mappings $f : A \to B$.

   b. Show that if $n > m$, there are no onto mappings $f : A \to B$.

   c. Assume $n \geq m$. How many different one-to-one mappings $f : A \to B$ are there? Prove your assertion.

3. Refer back to Problem 4 in the 'A' section for some examples of the patterns described in this exercise.

   a. Show that if f is not one-to-one, then there is some $S \subset A$ such that $f^{-1}(f(S)) \neq S$.

   b. Conversely, show that if f is one-to-one, then $f^{-1}(f(S)) = S$ for all $S \subset A$.

   c. Show that if f is not onto, then there is some $T \subset B$ such that $f(f^{-1}(T)) \neq T$

   d. Conversely, show that if f is onto, then $f(f^{-1}(T)) = T$ for all $T \subset B$.