Mathematics 243, section 3 – Algebraic Structures
Problem Set 1
due: September 7, 2012

'A' Section

1. Let

\[ U = \{1, 2, 3, \ldots, 12\} \]
\[ A = \{1, 3, 5, 7, 9, 11\} \]
\[ B = \{4, 6, 8\} \]
\[ C = \{7, 8, 9, 10\} \]

Find each of the following sets:

a. \( A' \cap C \)

b. \( A \cup C' \)

c. \( A \cap (B \cup C) \)

d. \( (A - B) \cup (B - C) \)

2. Let \( S = \{a, b, c, d\} \)

a. List all elements of \( P(S) \) (the power set)

b. Write out all of the partitions of \( S \).

3. Let \( A = \{1, 2\} \). For each pair of subsets \( S, T \subseteq A \) (including the cases where \( S \) or \( T \) is \( \emptyset \) or \( A \) itself), define \( S + T = (S \cup T) - (S \cap T) \). Make a chart with rows labeled with the possibilities for \( S \), columns labeled with the possibilities for \( T \), and entries showing which subset of \( A \) is produced as \( S + T \). (For example, in the row for \( S = \{1\} \) and the column for \( T = \{1, 2\} \), your chart will contain the entry \( S + T = \{1, 2\} - \{1\} = \{2\} \).

4. Let \( A = \{1, 2, 3, 4, 5\} \), \( B = \{a, b, c, d, e\} \) and let \( f : A \to B \) be defined by \( f(1) = a \), \( f(2) = e \), \( f(3) = c \), \( f(4) = a \), and \( f(5) = e \). Find the following:

a. \( f(A) \)

b. \( f(S) \) for \( S = \{3, 4, 5\} \subseteq A \)

c. \( f^{-1}(f(S)) \) for \( S = \{1, 2, 3\} \subseteq A \)

d. \( f^{-1}(T) \) for \( T = \{c, d, e\} \subseteq B \)

e. \( f(f^{-1}(T)) \) for \( T = \{a, b, c\} \subseteq B \)

5. Let \( P \) be the set of positive integers and let \( f : P \to P \) be defined by

\[
 f(x) = \begin{cases} 
 3x + 1 & \text{if } x \text{ is odd} \\
 x/2 & \text{if } x \text{ is even}
\end{cases}
\]
a. Compute \( f(7), f(f(7)), f(f(f(7))), \) etc. What happens eventually if you continue this long enough? Try the same thing with \( f(9), f(f(9)), f(f(f(9))), \) etc.

b. Show by example that \( f \) is \textit{not} one-to-one.

c. Explain why \( f \) \textit{is} onto.

'B' \textit{Section}

1. Let \( A, B \) be any two subsets of a universal set \( U \).
   a. Prove that \( (A \cap B)' = A' \cup B' \).
   b. Prove that \( (A \cup B) - (A \cap B) = (A \cap B') \cup (A' \cap B) \).

2. Let \( A \) and \( B \) be finite sets with \( |A| = m \) and \( |B| = n \).
   a. Show that if \( m > n \), there are no one-to-one mappings \( f : A \to B \).
   b. Show that if \( n > m \), there are no onto mappings \( f : A \to B \).
   c. Assume \( n \geq m \). How many different one-to-one mappings \( f : A \to B \) are there? Prove your assertion.

3. Refer back to Problem 4 in the 'A' section for some examples of the patterns described in this exercise.
   a. Show that if \( f \) is not one-to-one, then there is some \( S \subset A \) such that \( f^{-1}(f(S)) \neq S \).
   b. Conversely, show that if \( f \) is one-to-one, then \( f^{-1}(f(S)) = S \) for all \( S \subset A \).
   c. Show that if \( f \) is not onto, then there is some \( T \subset B \) such that \( f(f^{-1}(T)) \neq T \)
   d. Conversely, show that if \( f \) is onto, then \( f(f^{-1}(T)) = T \) for all \( T \subset B \).