1. Consider the following relations defined on the set $\mathbb{Z}$. In each case, say whether the relation is reflexive, symmetric, transitive. Justify your answers.

   a. $xRy$ if and only if $(-1)^x = (-1)^y$
      
      Solution: We see
      
      $$(-1)^x = \begin{cases} +1 & \text{if } x \text{ is even} \\ -1 & \text{if } x \text{ is odd} \end{cases}$$
      
      Hence $xRx$ is true for all $x$, so $R$ is reflexive. Similarly, if $xRy$ then $x, y$ are both even or both odd. So $yRx$ follows, and $R$ is symmetric. Finally $R$ is transitive since $xRy$ and $yRz$ imply $x, y, z$ are either all even or all odd.

   b. $xRy$ if and only if $x \cdot y \geq 0$
      
      Solution: $xRx$ is true for all $x \in \mathbb{Z}$, since $x^2 \geq 0$. So $R$ is reflexive. Similarly, $R$ is symmetric since multiplication in $\mathbb{Z}$ is commutative, so $xy = yx$ and if $xy \geq 0$, $yx \geq 0$ too. $R$ is not transitive since for instance $(1)R(0)$ is true and $(0)R(-1)$ is true, but $(1)R(-1)$ is not true.

   c. $xRy$ if and only if $|x - y| \leq 2$
      
      Solution: This is similar to b in the sense that this relation is reflexive since $|x - x| = 0 \leq 2$ for all $x$, and symmetric since $|x - y| = |y - x|$ for all $x, y \in \mathbb{Z}$. But it is not transitive, since for example $(-2)R(0)$ is true since $|(-2) - 0| = 2 \leq 2$ and $(0)R(2)$ is true since $|0 - 2| = 2 \leq 2$, but $(-2)R(2)$ is not true since $|-2 - 2| = 4$.

   d. $xRy$ if and only if $x$ has the same number of base 10 digits as $y$, ignoring signs of $x, y$.
      
      Solution: This is reflexive – every $x$ has the same number of base 10 digits as itself. Similarly, it is symmetric since if $x$ has the same number of digits as $y$, then $y$ has the same number of digits as $x$. It is also transitive, since if $x$ and $y$ have the same number of digits and $y$ and $z$ have the same number of digits, then so do $x$ and $z$.

   e. $xRy$ if and only if the sum of the base 10 digits of $x$ is the same as the sum of the base 10 digits of $y$, ignoring signs of $x, y$.
      
      Solution: This is reflexive, symmetric, and transitive as in part d (just replace “number of digits” by “sum of digits” everywhere).

2. Which of the relations in question 1 are equivalence relations? For those that are, say exactly which integers make up the equivalence class $[11]$ using correct set notation.
Solution: From the answers to question 1, the relation in a is an equivalence relation and

\[[11] = \{m \in \mathbb{Z} \mid m \text{ is odd}\}\]

The relation in d is also an equivalence relation. The equivalence class of [11] consists of all positive or negative integers that have the exactly two digits in their base 10 forms:

\[[11] = \{\pm 10, \pm 11, \pm 12, \ldots, \pm 99\}\]

Finally, the relation in part e is also an equivalence relation. The class of 11 consists of all numbers whose digits add up to 2:

\[[11] = \{\pm 2, \pm 20, \pm 200, \ldots, \pm 11, \pm 101, \pm 110, \ldots\}\]

(Note: there can be any number of 0 digits in these.)

3. Let \( R \) be the relation on \( \mathbb{Z} \) defined by \( xRy \) if and only if \( 4x - 15y \) is a multiple of 11. Show that \( R \) is an equivalence relation and describe all of the equivalence classes for \( R \).

Solution: For all integers \( x \), \( xRx \) is true since \( 4x - 15x = 11 \cdot (-x) \) is a multiple of 11. Hence \( R \) is reflexive. If \( xRy \), then \( 4x - 15y = 11k \) for some integer \( k \). But then \( 4y - 15x = -(4x - 15y) - 11x - 11y = 11(-k - x - y) \) is a multiple of 11, so \( yRx \) follows. Hence \( R \) is symmetric. Finally, if \( xRy \) and \( yRz \) then \( 4x - 15y = 11k \) and \( 4y - 15z = 11\ell \) for some integers \( k, \ell \). But then \( 4x - 15y + 4y - 15z = 11(k + \ell) \), so \( 4x - 15z = 11(k + \ell + y) \) is also a multiple of 11. It follows that \( xRz \) is true, so \( R \) is transitive. This shows that \( R \) is an equivalence relation.

The equivalence class of any \( x \in \mathbb{Z} \) is the set

\([x] = \{y \mid xRy\} = \{y \mid 4x - 15y = 11k, \text{ for some integer } k\}\)

It can be seen that there are only 11 different classes:

\([0] = \{\ldots, -22, -11, 0, 11, 22, \ldots\}\)
\([1] = \{\ldots, -21, -10, 1, 12, 23, \ldots\}\)
\vdots
\([10] = \{\ldots, -12, -1, 10, 21, 32, \ldots\}\)

4. Decide whether each of the following statements is true. For those that are true, give a short proof using the postulates for \( \mathbb{Z} \) given in §2.1 of the text. For those that are false, give a counterexample.

a. If \( xy = xz \) for integers \( x, y, z \), then \( y = z \).

Solution: This is false: let \( x = 0, y = 1, z = 2 \).

b. If \( x < y \), then \( x^2 < y^2 \).
Solution: This is also false: Let $x = -3$ and $y = 2$. Then $x < y$, but $x^2 > y^2$.

c. If $z - x < z - y$, then $y < x$.

Solution: This is true. If $z - y > z - x$, then $(z - y) - (z - x) \in \mathbb{Z}^+$. But that says $x - y \in \mathbb{Z}^+$, so by the definition of the order relation $x > y$.

'B' Section

1. In class we showed that the distinct equivalence classes of an equivalence relation $R$ on a set $A$ give a partition of $A$. Conversely, suppose $A = \bigcup_{\lambda \in \mathcal{C}} A_\lambda$ is a partition of $A$. Show that the relation $R$ on $A$ defined by $aRa'$ if and only if $a, a'$ are both elements of the same subset $A_\lambda$ is an equivalence relation.

Solution: Let $a \in A$. Then $a$ is contained in only one of the $A_\lambda$, since they form partition. That implies $aRa$ is true, and $R$ is reflexive. Next, suppose $xRy$. That means that $x$ and $y$ are in the same set in the partition, so it follows that $y$ and $x$ are also in the same set. Hence $yRx$ is also true and $R$ is symmetric. Finally, if $xRy$ and $yRz$ are both true then $x, y, z$ are all in the same set in the partition so $xRz$ is also true. This shows $R$ is transitive too, hence an equivalence relation.

2. In both parts of this problem, you will be working in $\mathbb{Z}$, using the postulates from §2.1

a. Show that if $x \cdot y = 0$, then $x = 0$ or $y = 0$. (Hint: Argue by contraposition. By the trichotomy postulate 4, if $x \neq 0$, then $x \in \mathbb{Z}^+$ or $-x \in \mathbb{Z}^+$, and the same is true for $y$.)

Solution: We want to show that if $x \neq 0$ and $y \neq 0$, then $xy \neq 0$. By postulate 4 in the book’s numbering, if $x \neq 0$, then $x \in \mathbb{Z}^+$ or $-x \in \mathbb{Z}^+$. Similarly, $y \in \mathbb{Z}^+$ or $-y \in \mathbb{Z}^+$. There are four possible combinations of statements that can be true here. If $x \in \mathbb{Z}^+$ and $y \in \mathbb{Z}^+$, then $xy \in \mathbb{Z}^+$ by postulate 4b. If $-x \in \mathbb{Z}^+$ and $y \in \mathbb{Z}^+$, then $(-x)y \in \mathbb{Z}^+$ by postulate 4b. But $(-x)y = -(xy)$ by Theorem 2.2, and then $xy \neq 0$. Similarly, $x \in \mathbb{Z}^+$ and $-y \in \mathbb{Z}^+$, then $x(-y) \in \mathbb{Z}^+$ by postulate 4b. But $x(-y) = (-y)x = -(yx)$ by Theorem 2.2, and then $yx = xy \neq 0$. Finally, $-x \in \mathbb{Z}^+$ and $-y \in \mathbb{Z}^+$, then $(-x)(-y) \in \mathbb{Z}^+$ by Exercise 5 in this section of the book (proof follows same idea as the proof $(-1)(-1) = 1$ done in class) and then $xy \neq 0$.

b. From part a, deduce the cancellation law in $\mathbb{Z}$: If $x \cdot y = x \cdot z$ and $x \neq 0$, then $y = z$.

Solution: If $x \cdot y = x \cdot z$, then $x \cdot (y - z) = 0$. We are assuming $x \neq 0$, so part a implies $y - z = 0$, and hence $y = z$.

3. Prove by mathematical induction:
a. For all \( n \in \mathbb{Z}^+ \),
\[
1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.
\]

\textit{Solution:} The base case here is \( n = 1 \). The formula is true in that case because \( 1^2 = 1 = \frac{1 \cdot 2 \cdot 3}{6} \). Now assume the formula is true for \( k \in \mathbb{Z} \), and consider the case \( n = k + 1 \). We have, using the induction hypothesis and some algebra:

\[
1^2 + 2^2 + \cdots + (k+1)^2 = (1^2 + 2^2 + \cdots + k^2) + (k+1)^2
\]
\[
= \frac{k(k+1)(2k+1)}{6} + (k+1)^2
\]
\[
= \frac{(k+1)(2k^2 + k + 6(k+1))}{6}
\]
\[
= \frac{(k+1)(2k^2 + 7k + 6)}{6}
\]
\[
= \frac{(k+1)(k+2)(2k+3)}{6},
\]

which is what we wanted to show since
\[
\frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.
\]

b. For all \( n \in \mathbb{Z}^+ \),
\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}
\]

\textit{Solution:} The base case here is \( n = 1 \). The formula is true in that case because \( \frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{1}{1+1} \). Now assume the formula is true for \( k \in \mathbb{Z} \), and consider the case \( n = k + 1 \). We have, using the induction hypothesis and some algebra:

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(k+1)(k+2)} = \left( \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} \right) + \frac{1}{(k+1)(k+2)}
\]
\[
= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}
\]
\[
= \frac{k(k+2) + 1}{(k+1)(k+2)}
\]
\[
= \frac{k^2 + 2k + 1}{(k+1)(k+2)}
\]
\[
= \frac{(k+1)^2}{(k+1)(k+2)}
\]
\[
= \frac{k+1}{k+2},
\]

which is what we wanted to show since
\[
\frac{k+1}{k+2} = \frac{k+1}{(k+1)+1}.
\]
c. For all \( n \geq 2 \), \( n^3 > 1 + 2n \).

*Solution:* The base case is \( n = 2 \), and the inequality is true in that case since \( 8 > 5 \). Now assume \( k^3 > 1 + 2k \) and consider \( (k+1)^3 = k^3 + 3k^2 + 3k + 1 \). By the induction hypothesis, we see \( (k+1)^3 > 1 + 2k + 3k^2 + 3k + 1 > 1 + 2(k+1) \), since \( 3k^2 + 3k + 1 > 3k + 1 > 2 \) for all \( k \geq 2 \).

d. If \( |A| = n \), then \( |\mathcal{P}(A)| = 2^n \). (Hint: For the induction step, let \( A = \{a_1, \ldots, a_k, a_{k+1}\} \). Every subset of \( A \) is of one of two types – the ones containing \( a_{k+1} \) and the ones not containing \( a_{k+1} \). Count the number of subsets of each type by using the induction hypothesis.)

*Solution:* When \( n = 0 \), \( A = \emptyset \) has exactly 1 subset, namely \( \emptyset \). Therefore \( |\mathcal{P}(A)| = 1 = 2^0 \). So the base case is established. Now assume that \( |\mathcal{P}(A)| = 2^k \) whenever \( |A| = k \) and consider \( A = \{a_1, \ldots, a_k, a_{k+1}\} \). Following the hint, note that

\[
\mathcal{P}(A) = S_1 \cup S_2
\]

where \( S_1 = \{T \subseteq A \mid a_{k+1} \in T\} \) and \( S_2 = \{T \subseteq A \mid a_{k+1} \notin T\} \). By their definitions, \( S_1 \cap S_2 = \emptyset \), so

\[
|\mathcal{P}(A)| = |S_1| + |S_2|.
\]

The subsets in \( S_2 \) are in one-to-one correspondence with the subsets of \( \{a_1, \ldots, a_k\} \). Hence \( |S_2| = 2^k \) by the induction hypothesis. Every subset \( T \) in \( S_1 \) contains \( a_{k+1} \), so it can be written as \( T = T' \cup \{a_{k+1}\} \), where \( T' \) is a subset of \( \{a_1, \ldots, a_k\} \). Distinct \( T' \)'s give distinct \( T \)'s so there are exactly as many elements of \( S_1 \) as subsets of \( \{a_1, \ldots, a_k\} \). By the induction hypothesis again, that number is \( 2^k \). Hence

\[
|\mathcal{P}(A)| = |S_1| + |S_2| = 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}.
\]

Therefore \( |\mathcal{P}(A)| = 2^n \) for all \( n \geq 1 \) by induction.

4. The binomial coefficients are the numbers

\[
\binom{n}{\ell} = \frac{n!}{\ell!(n-\ell)!}
\]

(where \( 0! = 1 \) by convention).

a. Show using the definition that for all \( \ell \) with \( 1 \leq \ell \leq n \),

\[
\binom{n}{\ell} + \binom{n}{\ell-1} = \binom{n+1}{\ell}
\]

*Solution:* We have

\[
\binom{n}{\ell} + \binom{n}{\ell-1} = \frac{n!}{\ell!(n-\ell)!} + \frac{n!}{(\ell-1)!(n-\ell+1)!}
\]

The common denominator for these two fractions is \( \ell!(n-\ell+1)! \), so adding we have

\[
\frac{n!}{\ell!(n-\ell)!} + \frac{n!}{(\ell-1)!(n-\ell+1)!} = \frac{n!(n-\ell+1+\ell)}{\ell!(n+1-\ell)!} = \frac{(n+1)!}{\ell!(n+1-\ell)!} = \binom{n+1}{\ell}.
\]
b. (The Binomial Theorem) Show by induction that for all \( n \geq 1 \),

\[
(a + b)^n = \sum_{\ell=0}^{n} \binom{n}{\ell} a^{\ell} b^{n-\ell}
\]

(that is, the numbers \( \binom{n}{k} \) are exactly the coefficients of the various terms \( a^k b^{n-k} \) appearing in the expansion of \( (a + b)^n \)).

**Solution:** The base case \( n = 1 \) follows since \( \binom{1}{0} = 1 = \binom{1}{1} \), and \( (a + b)^1 = a + b = \binom{1}{0} b + \binom{1}{1} a \).

Now assume the theorem has been proved for \( n = k \) and consider the case \( n = k + 1 \). We have by the induction hypothesis,

\[
(a + b)^{k+1} = (a + b)^k(a + b)
\]

\[
= \left( \sum_{\ell=0}^{k} \binom{k}{\ell} a^{\ell} b^{k-\ell} \right) (a + b).
\]

Expand the product on the last line using the distributive law and collect like terms. The coefficients of \( a^{k+1} \) and \( b^{k+1} \) are both equal to 1 by inspection. These match the formula to be proved since \( \binom{k+1}{k+1} = \binom{k+1}{0} = 1 \). Now assume \( 1 \leq \ell \leq k \). The \( a^\ell b^{k+1-\ell} \) term comes from

\[
b \cdot \left( a^\ell b^{k-\ell} \text{ term in first factor } \right) + a \cdot \left( a^{\ell-1} b^{k-(\ell-1)} \text{ term in first factor } \right).
\]

From above (by the induction hypothesis), the coefficient of this term is

\[
\binom{k}{\ell} + \binom{k}{\ell - 1} = \binom{k+1}{\ell},
\]

where the right side comes by applying part a of this question. Therefore

\[
(a + b)^{k+1} = \sum_{\ell=0}^{k+1} \binom{k+1}{\ell} a^{\ell} b^{k+1-\ell},
\]

and the result follows by induction.