Mathematics 243, section 3 – Algebraic Structures
Solutions for Problem Set 2
due: September 14, 2012

'A' Section

1. Let $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be the indicated functions. In each case, say whether $f, g, f \circ g, g \circ f$ are one-to-one (injective) or onto (surjective), both, or neither. Justify your answers.

   a. $f(x) = 3x$, $g(x) = 4 - x$

   Solution: f is one-to-one since $f(x) = 3x = 3x' = f(x')$ implies $3(x - x') = 0$ in $\mathbb{Z}$. Hence $x - x' = 0$, so $x = x'$. f is not onto because, for instance, there is no $x \in \mathbb{Z}$ such that $f(x) = 3x = 1$. g is one-to-one since $4 - x = 4 - x'$ implies $x = x'$. g is also onto, since given any integer y, $g(x) = 4 - x = y$ when $x = 4 - y$. $(f \circ g)(x) = 3(4 - x) = 12 - 3x$. This is one-to-one since $12 - 3x = 12 - 3x'$ implies $3(x - x') = 0$, so $x = x'$ as before. This function is not onto since there is no integer x such that $(f \circ g)(x) = 12 - 3x = 1$ (for instance). Finally $g \circ f$ is the function $(g \circ f)(x) = 4 - 3x$. This is one-to-one and not onto for reasons similar to those given for $f \circ g$.

   b. $f(x) = |x|$, $g(x) = \begin{cases} x & \text{if } x \text{ is even} \\ x - 1 & \text{if } x \text{ is odd} \end{cases}$.

   Solution: f is neither one-to-one nor onto, because $f(x) = f(-x)$ and f takes only nonnegative values. g is neither one-to-one nor onto. For instance $g(2) = 2 = g(3)$ so g is not one-to-one. g is not onto either because it takes only even values. $(f \circ g)(x) = \begin{cases} |x| & \text{if } x \text{ is even} \\ |x - 1| & \text{if } x \text{ is odd} \end{cases}$ is neither one-to-one nor onto since g is not one-to-one and f is not onto. $(g \circ f)(x) = \begin{cases} |x| & \text{if } x \text{ is even} \\ |x| - 1 & \text{if } x \text{ is odd} \end{cases}$ is neither one-to-one nor onto since f is not one-to-one and g is not onto.

2. Consider the binary operation on $\mathbb{Z}$ given by

$$x * y = x + y + 5$$

   a. Is $*$ commutative? Why or why not?

   Solution: Yes, since by commutativity of addition in $\mathbb{Z}$, $y * x = y + x + 5 = x + y + 5 = x * y$ is true for all $x, y \in \mathbb{Z}$.

   b. Is $*$ associative? Why or why not?

   Solution: We have

$$(x * y) * z = (x + y + 5) * z = (x + y + 5) + z + 5 = x + y + z + 10.$$

On the other hand,

$$x * (y * z) = x * (y + z + 5) = x + (y + z + 5) + 5 = x + y + z + 10.$$

Since these are the same for all $x, y, z \in \mathbb{Z}$, the operation is associative.

c. Is there an identity element for $*$. What is the identity, or why not?

Solution: Yes, $e = -5$ acts as an identity since $x * (-5) = x + (-5) + 5 = x$ and $(-5) * x = (-5) + x + 5 = x$ for all $x \in \mathbb{Z}$.

d. Are there any elements of $\mathbb{Z}$ that have inverses under this operation? What are they, and what are the inverses?

Solution: The inverse of x for this operation is the integer y that satisfies $x * y = -5$ (the identity from part c). Given x, $x + y + 5 = -5$ when $y = -x - 10$. so $-x - 10$ is the inverse of x under this operation.

3. Let $A = \{x, y, z, w\}$ and let $*$ be the binary operation on A given by the following table:

$*$	x	y	z	w
x	x	y	z	w
y	y	y	w	w
z	z	w	z	w
w	w	w	w	w

a. Explain how you can tell this operation is commutative.

Solution: The table is symmetric about the "main diagonal" from upper left to lower right. This means that $a * b = b * a$ for all $a, b \in A$, so the operation is commutative.

b. Explain why x is an identity element for $*$.

Solution: From the table, $a * x = x * a = a$ for all $a \in A$.

c. Which elements have inverses and what are the inverses?

Solution: x is the only element with an inverse, and the inverse is x itself.

d. What is $(y * z) * z$? Is that the same as $y * (z * z)$?

Solution: $(y * z) * z = w * z = w$. That is the same as $y * (z * z) = y * z = w$. (This, by itself, does *not* say that $*$ is associative, though. Do you see why not?)

'B' Section

1. Let $f : A \to B$ and $g : B \to A$ be mappings. Prove that if $f \circ g$ is onto and $g \circ f$ is one-to-one, then f is one-to-one *and* onto.

Solution: If $f \circ g$ is onto, then for every $b \in B$, there is some $x \in B$ such that $(f \circ g)(x) = b$. But that says $f(g(x)) = b$, so for every $b \in B$, there is some element in A (namely $g(x)$) such that $f(g(x)) = b$. This shows f is onto. For the other part we will prove the contrapositive form – If f is not one-to-one, then $g \circ f$ is not one-to=one either. If f is not one-to-one, then there exist $a \neq a'$ in A such that $f(a) = f(a')$. But then $g(f(a)) = g(f(a'))$ too, so $g \circ f$ is not one-to-one either.

2. Let $*$ be an associative binary operation on a set A and assume there is an identity element e for $*$. If $a \in A$ has inverses b_1 and b_2, show that $b_1 = b_2$. Hint: Consider the "product" $(b_1 * a) * b_2$.

   Solution: If b_1 is an inverse for $*$, then $(b_1 * a) * b_2 = e * b_2 = b_2$. But on the other hand, if $*$ is associative we also have $(b_1 * a) * b_2 = b_1 * (a * b_2)$. Then since b_2 is also an inverse for a, this equals $b_1 * e = b_1$. Two things that are equal to the same thing are equal to one another, so $b_1 = b_2$.

3. Let A be a set and let $\mathcal{P}(A)$ be the power set of A as defined in §1 of the text and on Problem Set 1. Let $*$ be the binary operation on $\mathcal{P}(A)$ defined by $S * T = S \cup T$. Answer the following questions and prove your assertions.

   a. Is $*$ associative? Is $*$ commutative?

   Solution: Yes to both. Commutativity just says $S * T = S \cup T = T \cup S = T * S$ and that follows from the definition of set union. Similarly, $*$ is associative since for any subsets S, T, U of A, $(S * T) * U = (S \cup T) \cup U$. This is the set of all elements of A, that are in S, or in T, or in U, which is the same as $S \cup (T \cup U) = S * (T * U)$.

   b. Is there an identity element in $\mathcal{P}(A)$ for this operation?

   Solution: Yes, $\emptyset$ (the empty subset of A) is an identity element, since $\emptyset \cup S = S \cup \emptyset = S$ for all $S \subseteq A$.

   c. What elements of $\mathcal{P}(A)$ have inverses for this operation?

   Solution: If $S = \emptyset$, then let $T = \emptyset$ too. Then $S \cup T = \emptyset = T \cup S$. Therefore $S = \emptyset$ does have an inverse. Now, we claim that this is the *only* subset of A that does have an inverse for this operation: if S does have an inverse, then $S = \emptyset$. We will show the contrapositive form. Let $S \neq \emptyset$. An inverse for S would be a subset T such that $S \cup T = \emptyset$. But $S \subseteq S \cup T$ for all T, so $S \cup T \neq \emptyset$. Therefore $S = \emptyset$ is the only S that does have an inverse.

   d. Make a table like the one in problem 3 of the 'A' section for the operation in this problem, when $A = \{a, b\}$. List the elements of $\mathcal{P}(A)$ in this order on the borders of the table:

$$\emptyset, \{a\}, \{b\}, \{a, b\}.$$

   Do you notice something?

   Solution: The table is:

$*$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$
$\emptyset$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$
$\{a\}$	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a,b\}$
$\{b\}$	$\{b\}$	$\{a,b\}$	$\{b\}$	$\{a,b\}$
$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$

   The thing you should notice is that this table has exactly the same "pattern" as the table from problem 3 in the 'A' section. If you replace $\emptyset \mapsto x$, $\{a\} \mapsto y$, $\{b\} \mapsto z$, $\{a, b\} \mapsto w$, then you get exactly the other table.

4. Let $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ be the vector *cross product* from multivariable calculus (MATH 241). Recall that this operation is defined by the following formula:

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2 b_3 - a_3 b_2, -(a_1 b_3 - a_3 b_1), a_1 b_2 - a_2 b_1).$$

a. Show that $\times$ is not associative and not commutative.

Solution: To show an operation does not have these properties, it suffices to find specific cases where they do not hold (negation of a "for all" statement is a "there exists" statement). Let $\mathbf{a} = (1, 0, 0)$, $\mathbf{b} = (0, 1, 0)$, $\mathbf{c} = (1, 1, 0)$ We have $\mathbf{a} \times \mathbf{b} = (0, 0, 1)$, but $\mathbf{b} \times \mathbf{a} = (0, 0, -1)$, so $\times$ is not commutative. Also,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (0, 0, 1) \times (1, 1, 0) = (-1, 1, 0).$$

But

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (1, 0, 0) \times (0, 0, 1) = (0, -1, 0).$$

Hence $\times$ is not associative.

b. Show that $\times$ does satisfy the *Jacobi identity*:

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a})$$

for all $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ in $\mathbb{R}^3$. The $+$ in this formula means the vector sum in $\mathbf{R}^3$, defined for vectors $\mathbf{d} = (d_1, d_2, d_3)$ and $\mathbf{e} = (e_1, e_2, e_3)$ by the rule

$$\mathbf{d} + \mathbf{e} = (d_1, d_2, d_3) + (e_1, e_2, e_3) = (d_1 + e_1, d_2 + e_2, d_3 + e_3).$$

Solution: Since this is a "for all" statement, it *does not* suffice just to give an example where the equation is true. Instead, we must show that the equation holds for all choices of vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ in $\mathbb{R}^3$. To see this, we compute as follows:

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} &= (a_2 b_3 - a_3 b_2, -(a_1 b_3 - a_3 b_1), a_1 b_2 - a_2 b_1) \times (c_1, c_2, c_3) \qquad (1)\\ &= ((a_3 b_1 - a_1 b_3) c_3 - (a_1 b_2 - a_2 b_1) c_2, (a_3 b_2 - a_2 b_3) c_3 + (a_1 b_2 - a_2 b_1) c_1,\\ &\quad (a_2 b_3 - a_3 b_2) c_2 + (a_1 b_3 - a_3 b_1) c_1)\\ &= (a_3 b_1 c_3 - a_1 b_3 c_3 - a_1 b_2 c_2 + a_2 b_1 c_2, a_3 b_2 c_3 - a_2 b_3 c_3 + a_1 b_2 c_1 - a_2 b_1 c_1,\\ &\quad a_2 b_3 c_2 - a_3 b_2 c_2 + a_1 b_3 c_1 - a_3 b_1 c_1).\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (a_1, a_2, a_3) \times (b_2 c_3 - b_3 c_2, -(b_1 c_3 - b_3 c_1), b_1 c_2 - b_2 c_1) \qquad (2)\\ &= (a_2 (b_1 c_2 - b_2 c_1) + a_3 (b_1 c_3 - b_3 c_1), -a_1 (b_1 c_2 - b_2 c_1) + a_3 (b_2 c_3 - b_3 c_2),\\ &\quad - a_1 (b_1 c_3 - b_3 c_1) - a_2 (b_2 c_3 - b_3 c_2))\\ &= (a_2 b_1 c_2 - a_2 b_2 c_1 + a_3 b_1 c_3 - a_3 b_3 c_1, -a_1 b_1 c_2 + a_1 b_2 c_1 + a_3 b_2 c_3 - a_3 b_3 c_2,\\ &\quad - a_1 b_1 c_3 + a_1 b_3 c_1 - a_2 b_2 c_3 + a_2 b_3 c_2)\end{aligned}$$

and

$$\begin{aligned}\mathbf{b}\times(\mathbf{c}\times\mathbf{a}) &= (b_1,b_2,b_3)\times(c_2a_3-c_3a_2,-(c_1a_3-c_3a_1),c_1a_2-c_2a_1) \\ &= (b_2(c_1a_2-c_2a_1)+b_3(c_1a_3-c_3a_1),-b_1(c_1a_2-c_2a_1)+b_3(c_2a_3-c_3a_2), \\ &\quad -b_1(c_1a_3-c_3a_1)-b_2(c_2a_3-c_3a_2)) \\ &= (a_2b_2c_1-a_1b_2c_2+a_3b_3c_1-a_1b_3c_3,-a_2b_1c_1+a_1b_1c_2+a_3b_3c_2-a_2b_3c_3, \\ &\quad -a_3b_1c_1+a_1b_1c_3-a_3b_2c_2+a_2b_2c_3)\end{aligned} \tag{3}$$

Adding $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})$, we see from Eqs. (2), (3) that there are cancellations in every component of the vectors on the right side. What is left is

$$\begin{aligned}&(a_2b_1c_2+a_3b_1c_3-a_1b_2c_2-a_1b_3c_3,a_1b_2c_1+a_3b_2c_3-a_2b_1c_1-a_2b_3c_3, \\ &\quad +a_1b_3c_1+a_2b_3c_2-a_3b_1c_1-a_3b_2c_2),\end{aligned}$$

which is the same as (1). This proves the Jacobi identity.

c. *Extra Credit* In a sense, the additional term $\mathbf{b}\times(\mathbf{a}\times\mathbf{c})$ on the right in the Jacobi identity measures the failure of associativity. Using that idea, is $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ *ever* equal to $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ when all three of the vectors are nonzero? Explain. Hint: One way to approach this is to think about the geometric conditions on the three vectors under which it will be true that

$$\mathbf{b}\times(\mathbf{c}\times\mathbf{a})=\mathbf{0}.$$

Solution: One condition under which associativity will hold is this: The cross product of the two vectors $\mathbf{c}$ and $\mathbf{c}$ is the zero vector $(\mathbf{c}\times\mathbf{a})=(0,0,0))$ when $\mathbf{c}$ and $\mathbf{a}$ point along the same line. If that is true then the associative law does hold for those $\mathbf{c}$ and $\mathbf{a}$ with any $\mathbf{b}$. There are other situations too, for instance if $\mathbf{b}$ points along the same line as $\mathbf{a}\times\mathbf{c}$.