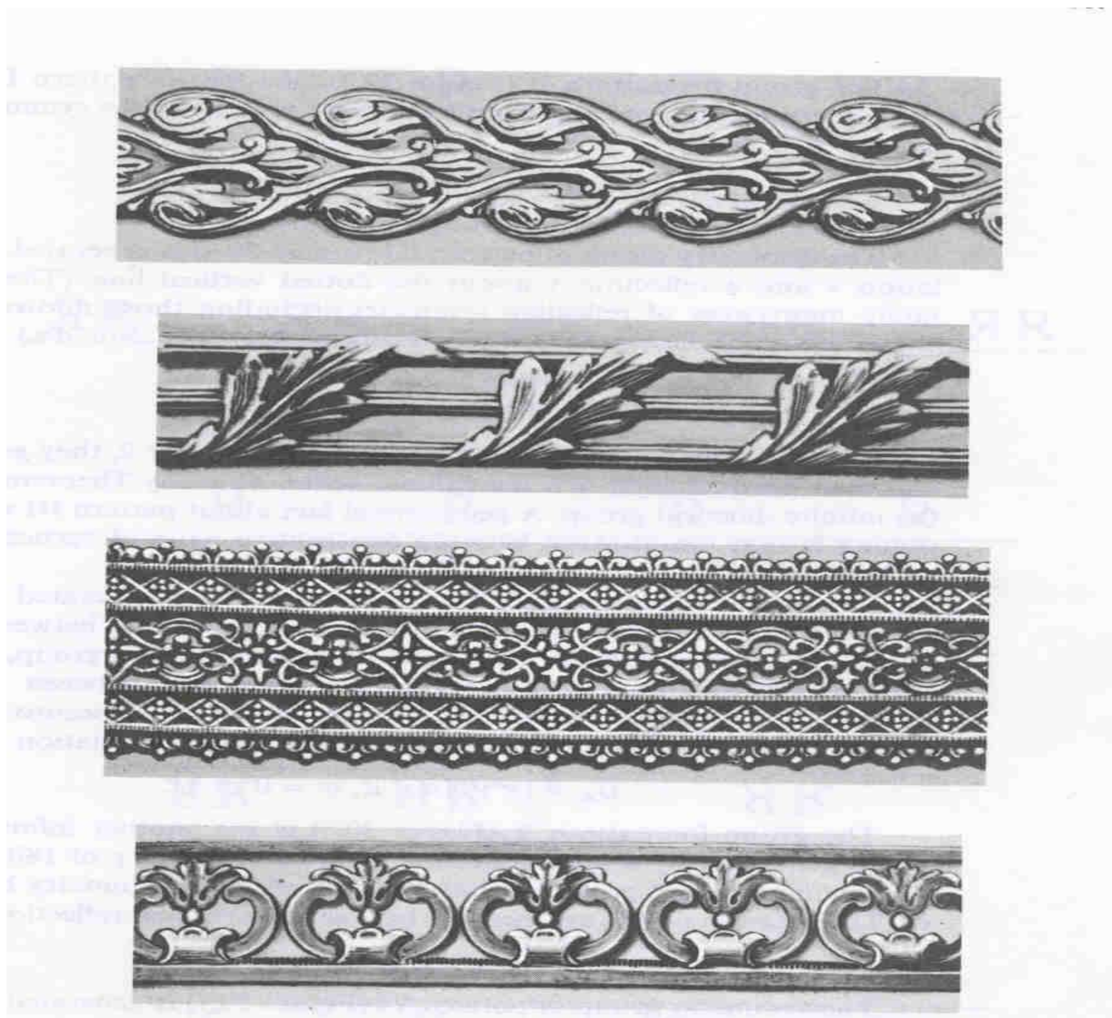
Algebraic Structures, December 4
Symmetry in art and design

- Many styles of architecture, design, and art use symmetry as a basic organizing tool.
- The mathematical notion of symmetry as *invariance under a transformation* can be used for analysis of how patterns are “put together.”
- The theory of groups we have studied is a key component of the mathematics involved.

Frieze and wallpaper patterns

- In textiles, jewelry, decorative work in interiors and exteriors of buildings, etc., repeating patterns of elements arranged along a line or in a strip are often used.
- These called *frieze patterns,* from a term used in architecture:
- *“frieze” = a long band of painted, sculpted, or calligraphic decoration (often placed above eye level)*

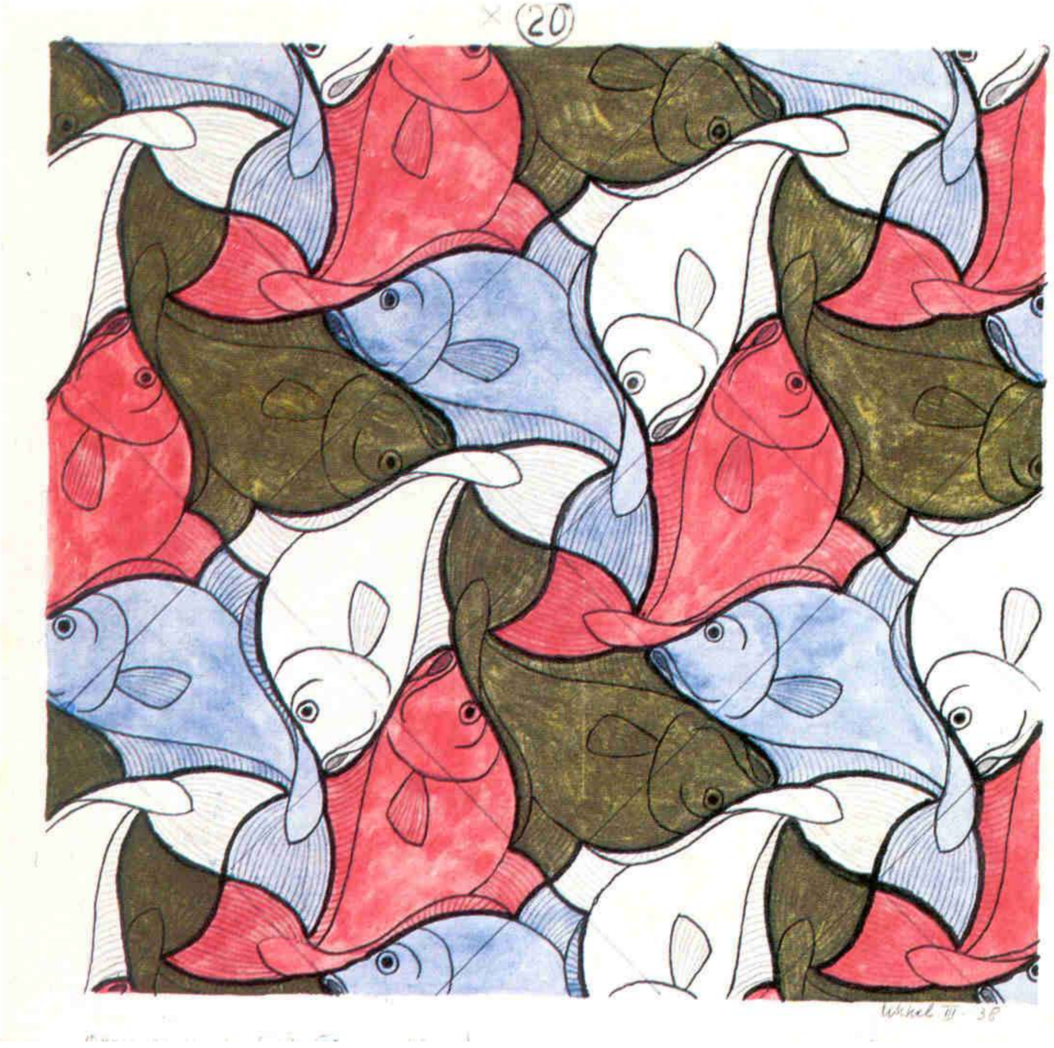
Some frieze patterns



Tesselations

- If a pattern is not limited to a single strip, but can “fill” an entire plane, it is called a “wallpaper pattern” or tesselation.
- Many interesting such patterns appear in the drawings and prints of the 20th century Dutch artist M.C. Escher.

An Escher “wallpaper” pattern



Symmetries in Escher drawing

- Imagine the Escher drawing extended to cover the whole plane ("idealized" version). Then, the whole pattern is invariant under:
  - *translations* by vectors in a 2-dimensional lattice $L = \{m\, v + n\, w : m, n \text{ in } Z\}$ (subdivides plane into squares in this case)
  - *rotations by 90 degrees* around the points in L and at centers of squares.
  - *rotations by 180 degrees* around the points such as (½) v, (½) w, and so forth.

The link to algebra

- Let G represent the collection of all distance-preserving mappings of the Euclidean plane to itself. *(G is a group under composition of mappings, generated by translations by all nonzero vectors, rotations about the origin, and reflection across the x-axis.)*
- If X is a figure in the plane, let $H(X)$ represent the collection of all elements T of G mapping X to itself (as a set).
- Then $H(X)$ is a subgroup of G, called the *group of symmetries* of X.

Group of symmetries

- Reason this is true:
  - The identity element of G is the identity mapping I on the plane, and this is in $H(X)$ since $I(X) = X$.
  - If S, T are elements of $H(X)$, they satisfy $S(X) = X$ and $T(X) = X$. The operation in G *is* composition of mappings so $(ST)(X) = S(T(X)) = S(X) = X$. This shows ST is in $H(X)$, so $H(X)$ is closed under the operation in G.
  - If T is in $H(X)$, and S is the inverse mapping of T (that is, $ST = I$), then $S(X) = X$ also, so $H(X)$ is also closed under inverses.

The Escher drawing again

- In fact the symmetry group of the Escher drawing is generated by the translations by two generators of the lattice L, and the 90 degree rotation around (0,0) in the plane.
- To see this:
  - In coordinates, say $v = (1,0)$, $w = (0,1)$.
  - Translation by v: $T(x,y) = (x + 1, y)$
  - *Rotation by 90 degrees about (0,0), counterclockwise:* $S(x,y) = (-y,x)$.

Escher symmetries explained

- Then, for instance, $TS(x,y) = (-y+1,x)$ gives rotation by 90 degrees about the point (½ , ½)
- Similarly, $STS(x,y) = (-x, -y+1)$ is the 180 degree rotation about (0, ½).
- "Moral:" All of the symmetries we see in the drawing are consequences of the translation invariance together with the rotation invariance for the rotation about one of the four-fold centers such as the corner (0,0) of the fundamental parallelogram.

Classification by symmetry group

- Mathematicians and crystallographers have developed a complete classification of frieze and wallpaper patterns based on the groups of symmetries.
  - There are exactly 7 different classes of frieze patterns
  - There are exactly 17 different classes of wallpaper patterns
- Technical note: the equivalence relation here is a bit finer than just isomorphism of groups (also uses the geometric nature of the symmetry transformations – distinguishes rotations by 180 degrees from reflections).

The 17 wallpaper groups

- Can you see where our Escher drawing fits?

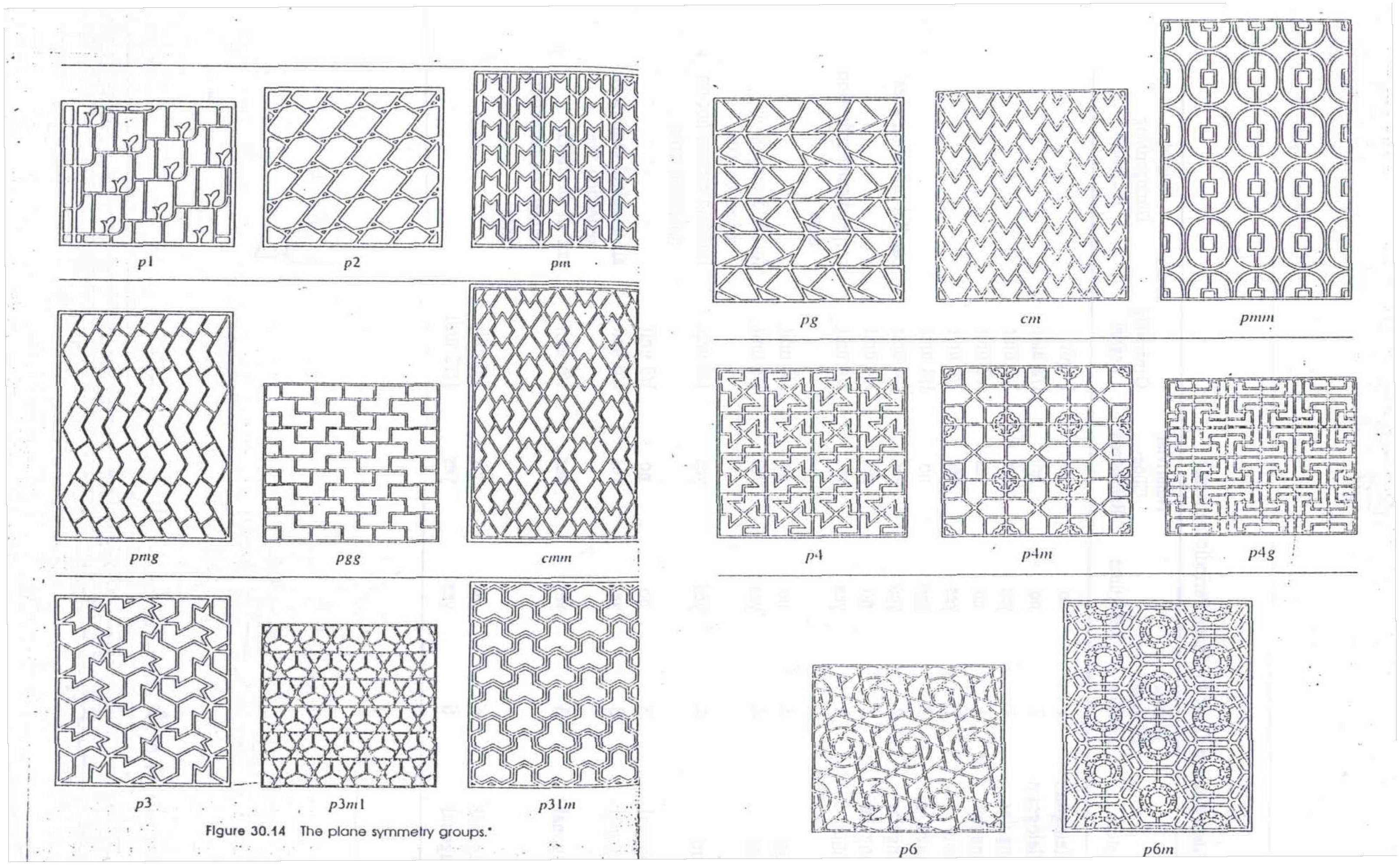


Figure 30.14 The plane symmetry groups.*