Symmetry in art and design

- Many styles of architecture, design, and art use symmetry as a basic organizing tool.
- The mathematical notion of symmetry as *invariance under a transformation* can be used for analysis of how patterns are “put together.”
- The theory of groups we have studied is a key component of the mathematics involved.
Frieze and wallpaper patterns

- In textiles, jewelry, decorative work in interiors and exteriors of buildings, etc., repeating patterns of elements arranged along a line or in a strip are often used.
- These called *frieze patterns*, from a term used in architecture:
  - “frieze” = a long band of painted, sculpted, or calligraphic decoration (often placed above eye level)
Some frieze patterns
Tesselations

- If a pattern is not limited to a single strip, but can “fill” an entire plane, it is called a “wallpaper pattern” or tesselation.
- Many interesting such patterns appear in the drawings and prints of the 20\textsuperscript{th} century Dutch artist M.C. Escher.
An Escher “wallpaper” pattern
Symmetries in Escher drawing

Imagine the Escher drawing extended to cover the whole plane ("idealized" version). Then, the whole pattern is invariant under:

- *translations* by vectors in a 2-dimensional lattice $L = \{m \ v + n \ w : m, n \text{ in } \mathbb{Z}\}$ (subdivides plane into squares in this case)
- *rotations by 90 degrees* around the points in $L$ and at centers of squares.
- *rotations by 180 degrees* around the points such as $(\frac{1}{2}) \ v$, $(\frac{1}{2}) \ w$, and so forth.
Let $G$ represent the collection of all distance-preserving mappings of the Euclidean plane to itself. ($G$ is a group under composition of mappings, generated by translations by all nonzero vectors, rotations about the origin, and reflection across the $x$-axis.)

If $X$ is a figure in the plane, let $H(X)$ represent the collection of all elements $T$ of $G$ mapping $X$ to itself (as a set).

Then $H(X)$ is a subgroup of $G$, called the \textit{group of symmetries} of $X$. 
Group of symmetries

Reason this is true:

- The identity element of $G$ is the identity mapping $I$ on the plane, and this is in $H(X)$ since $I(X) = X$.
- If $S$, $T$ are elements of $H(X)$, they satisfy $S(X) = X$ and $T(X) = X$. The operation in $G$ is composition of mappings so $(ST)(X) = S(T(X)) = S(X) = X$. This shows $ST$ is in $H(X)$, so $H(X)$ is closed under the operation in $G$.
- If $T$ is in $H(X)$, and $S$ is the inverse mapping of $T$ (that is, $ST = I$), then $S(X) = X$ also, so $H(X)$ is also closed under inverses.
The Escher drawing again

- In fact the symmetry group of the Escher drawing is generated by the translations by two generators of the lattice $L$, and the 90 degree rotation around (0,0) in the plane.

- To see this:
  - In coordinates, say $v = (1,0)$, $w = (0,1)$.
  - Translation by $v$: $T(x,y) = (x + 1, y)$
  - Rotation by 90 degrees about (0,0), counterclockwise: $S(x,y) = (-y,x)$. 
Then, for instance, $TS(x,y) = (-y+1,x)$ gives rotation by 90 degrees about the point $(\frac{1}{2}, \frac{1}{2})$.

Similarly, $STS(x,y) = (-x, -y+1)$ is the 180 degree rotation about $(0, \frac{1}{2})$.

“Moral:” All of the symmetries we see in the drawing are consequences of the translation invariance together with the rotation invariance for the rotation about one of the four-fold centers such as the corner (0,0) of the fundamental parallelogram.
Mathematicians and crystallographers have developed a complete classification of frieze and wallpaper patterns based on the groups of symmetries.

- There are exactly 7 different classes of frieze patterns
- There are exactly 17 different classes of wallpaper patterns

Technical note: the equivalence relation here is a bit finer than just isomorphism of groups (also uses the geometric nature of the symmetry transformations – distinguishes rotations by 180 degrees from reflections).
The 17 wallpaper groups

Can you see where our Escher drawing fits?