Mathematics 243 – Algebraic Structures
Selected Solutions – Problem Set 5
October 16, 2006

2.2/21. To show: $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$ for all $1 \le r \le n$. For this we do not need a proof by induction. We can simply use the definition of the binomial coefficients in terms of quotients of factorials, and simplify by putting terms over a common denominator:

$$\begin{aligned}\binom{n}{r} + \binom{n}{r-1} &= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!} \\ &= \frac{n!(n-r+1)}{r!(n-r+1)!} + \frac{n!r}{r!(n-r+1)!} \\ &= \frac{n!(n-r+1+r)}{r!(n-r+1)!} \\ &= \frac{n!(n+1)}{r!(n-r+1)!} \\ &= \frac{(n+1)!}{r!(n+1-r)!} \\ &= \binom{n+1}{r}\end{aligned}$$

2.2/23. We want to prove the binomial theorem:

$$(a+b)^n = \sum_{\ell=0}^{n} \binom{n}{\ell} a^{n-\ell} b^{\ell}$$

for all $n \ge 1$. (The formulas actually work with $n = 0$ also.)

We will take the base case as $n = 1$. Then $\binom{1}{0} = \frac{1!}{0!1!} = 1$ and $\binom{1}{1} = \frac{1!}{1!0!} = 1$. Hence

$$(a+b)^1 = a + b = \binom{1}{0} a + \binom{1}{1} b$$

is true.

Now assume that the theorem is true for $n = k$:

$$(a+b)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} a^{k-\ell} b^{\ell}$$

and consider the case $n = k+1$. We will start from $(a+b)^{k+1}$ and apply the induction hypothesis:

$$\begin{aligned}(a+b)^{k+1} &= (a+b)^k (a+b) \\ &= \left(\sum_{\ell=0}^{k} \binom{k}{\ell} a^{k-\ell} b^{\ell}\right)(a+b) \\ &= \left(\binom{k}{0} a^k + \sum_{\ell=1}^{k-1} \binom{k}{\ell} a^{k-\ell} b^{\ell} + \binom{k}{k} b^k\right)(a+b)\end{aligned}$$

Expand out the product completely using the distributive law, and collect like terms. The "end terms" are $\binom{k}{0}a^k \cdot a = a^{k+1} = \binom{k+1}{0}a^{k+1}$ and $\binom{k}{k}b^k \cdot b = b^{k+1} = \binom{k+1}{k+1}b^{k+1}$ (since $\binom{k+1}{0} = \binom{k+1}{k+1} = 1$). In all other cases, for each given ℓ, $1 \le \ell \le k$, there are two different products that yield the same $a^{k+1-\ell}b^\ell$, namely:

$$\binom{k}{\ell-1}a^{k-(\ell-1)}b^{\ell-1} \cdot b + \binom{k}{\ell}a^{k-\ell}b^\ell \cdot a = \left(\binom{k}{\ell-1} + \binom{k}{\ell}\right) a^{k+1-\ell}b^\ell$$

By the result of problem 21, the sum of binomial coefficents in the parentheses here is $\binom{k+1}{\ell}$ so the product

$$\left(\sum_{\ell=0}^{k} \binom{k}{\ell} a^{k-\ell}b^\ell\right)(a+b) = \sum_{\ell=0}^{k+1} \binom{k+1}{\ell} a^{k+1-\ell}b^\ell,$$

which is what we wanted to show.

2.3/20. We want to show that if a, b are integers such that $a|b$ and $b|a$, then $a = b$ or $a = -b$. The hypothesis implies that there are integers p, q such that

$$(1) \qquad a = bp \quad \text{and} \quad b = aq.$$

If either $a = 0$ or $b = 0$ then both $a, b = 0$, and the conclusion follows since $0 = \pm 0$. Hence we may assume $a, b \neq 0$ in the remainder of the proof. If we substitute from the first equation in (1) above into the second, then we get the equation $b = (bp)q = b(pq)$. Since we are assuming $b \neq 0$, this implies $pq = 1$. But the only two ways two integers p, q can multiply to give 1 are $p = q = 1$ or $p = q = -1$. In the first case, $a = b$. In the second, $a = -b$. This is what we wanted to show.