Mathematics 243 – Algebraic Structures
Problem Set 2 – Selected Solutions
September 18, 2006

Section 1.3/

7. a) Let $x \in A$. Since g is onto, for all $y \in A$ there exists $x \in A$ such that $g(x) = y$, but then

$$\begin{aligned} f(y) &= f(g(x)) \\ &= (f \circ g)(x) \quad \text{by def. of composition} \\ &= (h \circ g)(x) \quad \text{since } f \circ g = h \circ g \quad \text{(hypothesis)} \\ &= f(h(x)) \quad \text{by def. of composition} \\ &= f(y) \end{aligned}$$

Since $f(y) = h(y)$ for all $y \in A$, the mappings are equal.

b) For all $x \in A$, we have $(f \circ g)(x) = (f \circ h)(x)$ by hypothesis. So $f(g(x)) = f(h(x))$. Since f is one-to-one, this implies $g(x) = h(x)$. Since this is true for all $x \in A$, the mappings are equal.

For the next two problems, we assume

$$g : A \to B \qquad \text{and} \qquad f : B \to C.$$

10. We are given that $f \circ g$ is onto, so for all $z \in C$, there exists some $x \in A$ such that

$$z = (f \circ g)(x) = f(g(x)).$$

But then for all $z \in C$, there exists some $y \in B$, namely $y = g(x)$, such that $z = f(y)$. This shows that f is onto.

11. To show that g is one-to-one, start by assuming $g(x) = g(x')$ for $x, x' \in A$. Then "plugging both sides into" f yields $f(g(x)) = f(g(x'))$. But now we are given $f \circ g$ is one-to-one, so this implies $x = x'$. Therefore g is also one-to-one.

13. We have, by the definition of the inverse image of a subset,

$$\begin{aligned} x \in (f \circ g)^{-1}(S) &\Leftrightarrow (f \circ g)(x) \in S \\ &\Leftrightarrow f(g(x)) \in S \\ &\Leftrightarrow g(x) \in f^{-1}(S) \\ &\Leftrightarrow x \in g^{-1}\left(f^{-1}(S)\right). \end{aligned}$$

Hence, $(f \circ g)^{-1}(S) = g^{-1}\left(f^{-1}(S)\right)$ as claimed.