Mathematics 243 – Algebraic Structures
Discussion 1 – Binary Operations
September 13, 2006

Background

Last time we introduced the idea of a *binary operation* on a set A, that is a mapping:

$$* : A \times A \to A$$
$$(a, a') \mapsto a * a'$$

We said

- $*$ is *associative* if (and only if) $(a * a') * a'' = a * (a' * a'')$ for all $a, a', a'' \in A$.
- $*$ is *commutative* if (and only if) $a * a' = a' * a$ for all $a, a' \in A$.
- An element $e \in A$ is an *identity element for* $*$ if $e * a = a$ and $a * e = a$ for all $a \in A$.

One additional concept:

- If $*$ is a binary operation on A that has an identity element e (this is necessary for what follows), then we say $b \in A$ is an *inverse* of a if $a * b = e$ and $b * a = e$.

Today, we want to practice with several additional examples of binary operations and study these concepts in more detail.

Discussion Questions

A) Let $* : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ be the binary operation defined for $x, y \in \mathbf{Z}$ by

$$x * y = x + y - xy.$$

Show (i.e. prove) the following:

1) $*$ is commutative
2) $*$ is associative
3) $*$ has an identity element (you'll need to find one, then show that it "works" in the definition).
4) There are only two elements of $\mathbf{Z}$ that have inverses for this operation (you'll need to find each of them and their inverses, say why they "work", then show that your elements are the only ones).

B) Let A be the set $\{a, b, c, d, f\}$ and consider the binary operation defined by the following table:

$*$	a	b	c	d	f
a	f	d	c	b	a
b	d	c	b	a	f
c	c	b	a	f	d
d	b	a	f	d	c
f	a	f	d	c	b

1) Is the operation $*$ commutative? Why or why not?
2) Does the operation $*$ have an identity element? Why or why not?
3) Is the operation $*$ associative. Why or why not?
4) What is true about $x^3 = x * x * x$ for all $x \in A$?

C) Let $*$ be a binary operation on a set A. Suppose e_1 is an identity element for $*$. Can there be *another* element $e_2 \neq e_1$ which is *also* an identity element for $*$. Prove your assertion.

Assignment

Writeups (one per group) due Monday, September 18.