Mathematics 243, section 1 – Algebraic Structures
Selected Solutions – Problem Set 7
November 20, 2003

3.4/1. Let $\phi : G \to H$ be a group isomorphism. Then we know ϕ is 1-1, onto and satisfies the group homomorphism property $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in G$. Since ϕ is 1-1 and onto, there exists an *inverse mapping* $\phi^{-1} : H \to G$. Since $\phi = (\phi^{-1})^{-1}$, ϕ^{-1} is also one-to-one and onto. Hence we only need to show that ϕ^{-1} is also a group homomorphism. Let $w, z \in H$. Since ϕ is onto, there exist $x, y \in G$ such that $\phi(x) = w$ and $\phi(y) = z$ (so $\phi^{-1}(w) = x$ and $\phi^{-1}(z) = y$. Hence

$$wz = \phi(x)\phi(y) = \phi(xy) \Rightarrow \phi^{-1}(wz) = xy = \phi^{-1}(w)\phi^{-1}(z)$$

This is what we wanted to show.

3.4/2. Although the notation $\phi_2\phi_1$ is slightly ambiguous in the problem, the only consistent interpretation from the context is that this is the *composite mapping*:

$$\phi_2 \circ \phi : G_1 \to G_3$$

From work in Chapter 1, we know that the composition of 1-1 mappings is 1-1 and the composition of onto mappings is onto. Hence since ϕ_1 and ϕ_2 are group isomorphisms, $\phi_2 \circ \phi_1$ is also 1-1 and onto. Then, if $x, y \in G_1$ are arbitrary, we have

$$(\phi_2 \circ \phi_1)(xy) = \phi_2(\phi_1(xy)) = \phi_2(\phi_1(x)\phi_1(y)) = \phi_2(\phi_1(x))\phi_2(\phi_1(y))$$

since both ϕ_1 and ϕ_2 have the group homomorphism property. It follows that $\phi_2 \circ \phi_1$ is also an isomorphism of groups.

3.4/21. Let G and H be any two cyclic groups of order n. By definition, this means that $G = \langle a \rangle$, $H = \langle b \rangle$ for some $a \in G$ and $b \in H$, and moreover, $a^n = b^n = e$ and n is the smallest strictly positive integer with this property. Therefore,

$$G = \{e = a^0, a = a^1, \ldots, a^{n-1}\}$$
$$H = \{e = b^0, b = b^1, \ldots, b^{n-1}\}$$

Define $\phi : G \to H$ by $\phi(a^k) = b^k$ for all $k \in \mathbf{Z}$. Then ϕ is clearly one-to-one and onto since it maps distinct elements of G to distinct elements of H and every element of H is the image of some element of G. Finally, letting $x = a^\ell$ $y = a^m$ be any two elements of G, we have

$$\phi(a^\ell \cdot a^m) = \phi(a^{\ell+m}) = b^{\ell+m} = b^\ell \cdot b^m = \phi(a^\ell)\phi(a^m)$$

so ϕ is an isomorphism of groups.

3.5/10. Let H be a homomorphic image of G (that is, there exists an onto group homomorphism $\phi : G \to H$). Let w, z be arbitrary elements of H. Then since ϕ is onto, $w = \phi(x)$ and $z = \phi(y)$ for some x, y in G. We have, using commutativity in G:

$$wz = \phi(x)\phi(y) = \phi(xy) = \phi(yx) = \phi(y)\phi(x) = zw$$

Hence H is also abelian.