Mathematics 243, section 1 – Algebraic Structures
Selected Solutions, Problem Set 3
October 1, 2003

1.4/9 Let $a \in A$, and assume b, c are both inverses of a for $*$. Consider the "product" $(b * a) * c$. First, since b is an inverse for a, $(b * a) * c = e * c = c$ (where the last equality comes since e is the identity element for $*$). But we also know $*$ is associative, so $(b * a) * c = b * (a * c) = b * e = b$. Hence $b = c$, so the inverse of a is unique (if it exists).

1.4/13 The problem says show that $f : A \to A$ has a left inverse (for composition) if and only if $f^{-1}(f(S)) = S$ for all subsets $S \subseteq A$. From Discussion 2, or Lemma 1.23 in the text, we know that f has a left inverse if and only if f is injective. Hence it suffices to prove that f is injective if and only if $f^{-1}(f(S)) = S$ for all $S \subseteq A$.

$\Rightarrow$: We show that f injective implies $f^{-1}(f(S)) = S$ for all S. The inclusion $S \subseteq f^{-1}(f(S))$ is automatic from the definitions of the direct and inverse images (if you don't see why, work this out from the definitions(!)). So the key point is the other inclusion. Let $x \in f^{-1}(f(S))$. By definition, this means $f(x) \in f(S)$, so $f(x) = f(a)$ for some $a \in S$. But we are assuming f is injective, so $x = a \in S$. Hence $x \in S$ and $f^{-1}(f(S)) \subseteq S$, which finishes this part of the proof.

$\Leftarrow$: Now we assume that $f^{-1}(f(S)) = S$ for all S, and use that to show f is injective. Let $x \in A$, and consider the subset $S = \{x\}$. By hypothesis in this part, $f^{-1}(f(\{x\})) = \{x\}$. If $f(a) = f(x)$ for some $a \in A$, then by the definition of the direct image, $f(a) \in f(S)$. Then by the definition of the inverse image, $a \in f^{-1}(f(S))$. But then $a \in S = \{x\}$, so the only possibility is $a = x$. Hence f is injective. This concludes the proof.

1.4/13 The problem says show that $f : A \to A$ has a right inverse (for composition) if and only if $f(f^{-1}(T)) = T$ for all subsets $T \subseteq A$. From Discussion 2, or Lemma 1.24 in the text, we know that f has a right inverse if and only if f is surjective. Hence it suffices to prove that f is surjective if and only if $f(f^{-1}(T)) = T$ for all $T \subseteq A$.

$\Rightarrow$: We show that f surjective implies $f(f^{-1}(T)) = T$ for all T. The inclusion $f(f^{-1}(T)) \subseteq T$ is automatic from the definitions of the direct and inverse images (if you don't see why, work this out from the definitions(!)). So the key point is the other inclusion. Let $x \in T$. Since f is surjective, there is some $a \in A$ such that $f(a) = x$. By definition $a \in f^{-1}(T)$, so $x = f(a) \in f(f^{-1}(T))$. Hence $T \subset f(f^{-1}(T))$.

$\Leftarrow$: Now we assume that $f(f^{-1}(T)) = T$ for all subsets, and use that to show f is surjective. As in problem 13 above, the "action" all goes on with one-element subsets(!) Let $x \in A$ be arbitrary, and consider $T = \{x\}$. Since $T = f(f^{-1}(T))$, $x = f(a)$ for some $a \in f^{-1}(T)$. But x was arbitrary in A, so this shows that all x in A are in the range of f. Hence f is surjective. This concludes the proof.

1.6/19. Let $\{A_\lambda\}$, $\lambda \in \mathcal{L}$ be a collection of subsets of A defining a partition of A. That is we are given:

$$\begin{cases} A = \bigcup_{\lambda \in \mathcal{L}} A_\lambda & \text{and} \\ A_\lambda \cap A_\mu = \emptyset & \text{if } \lambda \neq \mu \end{cases}$$

The problem is to show that if we define a relation R on A by

$$xRy \Leftrightarrow x, y \in \text{the same } A_\lambda$$

then R is an equivalence relation. (Note: by the second property of a partition, x and y cannot be in more than one of the A_λ.) To show that R is an equivalence relation, we check that R is reflexive, symmetric, and transitive.

reflexive: xRx is true for all x since there is just one subset A_λ such that $x \in A_\lambda$ (the different subsets in the partition do not overlap).

symmetric: If xRy is true, then $x, y \in A_\lambda$. But that says $y, x \in A_\lambda$ also, so yRx follows.

transitive: If xRy and yRz, then x, y are in the same A_λ, and y, z are in the same A_λ. Hence x, z are in the same A_λ, so xRz follows.