Mathematics 243, section 1 – Algebraic Structures
Information on Exam 3
November 24, 2003

General Information

The third exam this semester will be given the Friday after Thanksgiving, December 5. The exam will cover the material we have discussed since the second exam, through Cayley's Theorem from class on Monday, November 24 – the material from Chapters 3, 4 (Groups). in Gilbert and Gilbert. The topics to review are:

1) The definition of a group; how to determine if a given set with a given binary operation has the structure of a group; key examples such as $\mathbf{Z}_m, +$, the group of invertible classes in $\mathbf{Z}_m$ under multiplication: $\mathbf{Z}_m^\times, \cdot$, the group of invertible 2×2 matrices, $GL(2, \mathbf{R})$ under the matrix product, groups of permutations, etc., what it means for a group to be abelian, which of the key examples are abelian and which are not.
2) Subgroups, the general subgroup criterion from Theorem 3.9, cyclic subgroups.
3) Cyclic groups, generators, orders of elements, the distinct subgroups of a cyclic group of order n.
4) Isomorphisms of groups.
5) Group homomorphisms, the kernel, other properties.
6) Permutation groups, cycle decompositions, even and odd permutations, Cayley's Theorem.

Proofs to Know

1) Know how to prove Theorem 3.9 and how to apply it to show that subsets of groups are or are not subgroups, both in explicit examples and in cases where the subset is defined by a condition such as we saw in the problems about the center of a group, the conjugate subgroups aHa^{-1}, the kernel of a homomorphism, etc.
2) Let $G = \langle a \rangle$ be a cyclic group. Then
   a) Every subgroup H of G is cyclic.
   b) If G is finite cyclic of order n and $H = \langle a^k \rangle$, then $H = \langle a^d \rangle$ where $d = \gcd(n, k)$.
   c) If G is finite cyclic of order n, then a^k is a generator of G if and only if $\gcd(k, n) = 1$.
3) If $\varphi : G \to H$ is a group homomorphism, then $\varphi(e_G) = e_H$, and for all $x \in G$, $\varphi(x^{-1}) = (\varphi(x))^{-1}$.
4) The kernel of a group homomorphism is a subgroup of the domain.

Some Review Problems

From Gilbert and Gilbert:

1) Section 3.1: problems like 1-33, 44, 48, 49.

2) Section 3.2: 9, 10, 12, 13, 16, 23, 24, 35 ("closed" means closed under the group operation from G)
3) Section 3.3: 7, 8, 9, 11, 19, 20, 26, 27, 29, 30, 33
4) Section 3.4: 11, 16,17, 18
5) Section 3.5: 5,7,11,12
6) Section 4.1: problems like 1-9, 15
7) Section 4.2: 5

Review Session

I have off-campus commitments in Boston on the evenings of Wednesday, December 3, and Thursday, December 4. I will need to leave campus at 5:30 pm at the latest each of those days. Thus, if we are going to do an evening review session this time, it has to be Monday, December 1 or Tuesday, December 2. I am happy to do a session either evening, but be aware that the better prepared you are that evening, the more valuable the session will be. *A word to the wise: Do not put off studying for this exam until after the review session!*

Sample Exam

I. Let $\mathbf{Q}$ be the set of rational numbers: $\mathbf{Q} = \{m/n : m, n \in \mathbf{Z}, n \neq 0\}$. Define a binary operation $*$ on $\mathbf{Q}$ by $x * y = x + y + x \cdot y$ (where $\cdot$ is ordinary multiplication). Is $\mathbf{Q}$ a group under $*$? Why or why not?

II. Find all generators the group $G = \mathbf{Z}_{21}$, in which the operation is addition mod 21.

III. Is the symmetric group S_3 cyclic? Why or why not?

IV. Let $G = \langle a \rangle$ be a cyclic group.

A) Show that every subgroup $H \subset G$ is cyclic.

B) Show that if G is finite, with $|G| = n$, then $\langle a^k \rangle = \langle a^d \rangle$ where $d = \gcd(n, k)$.

V. Let $G = \mathbf{Z}_{12}$ and $H = \mathbf{Z}_9$, which are both groups under addition. We write $[x]_{12}$ for the congruence class of x mod 12, and similarly $[x]_9$ for the class mod 9. Define $\phi : G \to H$ by $\phi([x]_{12}) = [3x]_9$.

A) Show that $[x]_{12} = [y]_{12}$ implies $[3x]_9 = [3y]_9$ (so that this mapping actually makes sense).

B) Show that ϕ is a *group homomorphism.*

C) Find all the elements in $\ker(\phi)$.

VI. Let G be a group and let $a \in G$ be a fixed element. Define

$$C(a) = \{x \in G : ax = xa\}$$

A) Is $b = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ in $C(a)$ for $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $G = GL(2, \mathbf{R})$ (a group under matrix multiplication)? Why or why not?

B) Show that $C(a)$ is a subgroup of G.

Extra Credit (10) Can a group G with 5 elements contain an element a of order 3? Why or why not?