Mathematics 243, section 1 – Algebraic Structures
Discussion 2 – Composition Inverses
September 19, 2003

Background

Last time we discussed the binary operation of composition $\circ$ on the set $A = F(\mathbf{R}) = \{f : f : \mathbf{R} \to \mathbf{R}\}$. We also introduced the following general terminology for binary operations which have identity elements:

- We say $b \in A$ is a *left inverse of* $a \in A$ if $b * a = e$ (the identity element).
- We say $b \in A$ is a *right inverse of* $a \in A$ if $a * b = e$.
- We say $b \in A$ is an *inverse of* $a \in A$ if $a * b = b * a = e$.

Today, we want to tie these notions together with some of the function concepts (ranges, injectivity, surjectivity, etc.) that we learned last week.

Note: The facts we will be considering today are covered in the text. You may refer to the discussions in section 1.3 after class if you need to, but at least for now, try to "figure this out on your own" – it will be very good practice!

Discussion Questions

A) Of the following four functions in $F(\mathbf{R})$, one has neither a left nor a right (composition) inverse in $F(\mathbf{R})$, one has a left inverse but no right inverse, one has a right inverse but no left inverse, and one has a left and a right inverse (which are equal). Explain which is which and give formulas for the left or right inverse functions that do exist. *Note*: Your left and right inverse functions should be in $F(\mathbf{R})$ – that is the *domains should be the whole set of real numbers* in all cases.
   1) $f(x) = \arctan(x)$
   2) $g(x) = x^4$
   3) $h(x) = \sqrt[5]{x+7}$
   4) $i(x) = \begin{cases} \ln(|x|) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

B) If a mapping $f : A \to A$ has a right inverse $g : A \to A$ for composition, then $f(g(a)) = a$ for all $a \in A$. In particular, note that this says f is *surjective*. Say why this is true, and show this observation agrees with your answers in question A (with $A = \mathbf{R}$ for those functions).

C) Now, suppose we know that $f : A \to A$ is surjective. We can show that in fact f must have a right composition inverse as follows. Since f is surjective, for all $b \in A$, there exist some $a \in A$ such that $f(a) = b$. There may in fact be more than one such a, but just choose any one that "works" and let $g(b) =$ that a. Show that the mapping defined this way is a right composition inverse for f.

D) Saying that a mapping $f : A \to A$ has a left inverse for composition says that there is some mapping $g : A \to A$ such that $g \circ f = I_A$ (the identity function on A). If you have an equation $f(a) = f(b)$ for some $a, b \in A$ and you apply the left inverse g to both sides, what happens? What does this imply about f? Does this agree with what you saw in question A?

E) Your answer to question B should be equivalent to an implication f has a left composition inverse $\Rightarrow f$ has some other "standard" property we have discussed. The converse statement: f has the "standard property" $\Rightarrow f$ has a left composition inverse is also true. Say how to construct a left inverse if f has the "standard property." (Hint: There is no choice for where the left inverse maps $b \in \text{range}(f) \subseteq A$. The choice of what the inverse does on the complement of the range is actually arbitrary, though. Do you see why? Think about function $i(x)$ from part A.)

Assignment

Group write-ups due in class Wednesday, September 24.