Mathematics 243, section 1 – Algebraic Structures
Discussion 1: More on Mappings
September 12, 2003

Background

Last time, we introduced the following terminology for mappings (functions) $f : A \to B$:

1) The *domain* of f is the set A.
2) The *codomain* of f is the set B.
3) The *range* of f is the set $f(A) = \{b \in B | b = f(a) \text{ for some } a \in A\}$.
4) The function f is *surjective, (or onto)* if $f(A) = B$ (or equivalently, if for every $b \in B$, there exists some $a \in A$ such that $f(a) = b$).
5) The function f is *injective, or (one-to-one)* if $f(a) = f(a')$ implies $a = a'$ (or stated yet another way, if distinct elements $a \neq a'$ of A map to distinct elements $f(a) \neq f(a')$ of B).

Today we will practice using these ideas with some specific examples of functions. Before we begin, one comment about the definitions of the terms "surjective" and "injective" is probably in order.

Important Comment: It is conventional in making definitions of mathematical concepts or categories of objects to state the definition in the form: "x is a *foo* if it has the property *bar*". This is one case where the intended meaning is *not given by a straight translation* into symbolic logical form. The symbolic form would be

$$x \text{ has property } bar \Rightarrow x \text{ is a } foo$$

But that is **not** how you should read the definition. The definition really means:

$$x \text{ has property } bar \Leftrightarrow x \text{ is a } foo$$

In other words, for example, the definition of the surjectivity property for functions really means *both*: "if $f(A) = B$, then f is surjective" *and* "if f is surjective, then $f(A) = B$". Why do you suppose people do not include the other implication explicitly?

Discussion Questions

A) For each of the following functions from $\mathbf{R}$ to $\mathbf{R}$, the domain is the largest set of $x \in \mathbf{R}$ for which the formula makes sense. Say what the domain is, what the range is, and determine whether f is injective, surjective, both, or neither.

   1) f defined by $f(x) = x^2 + 5x + 6$
   2) g defined by $g(x) = \sqrt{1 - x^2}$
   3) h defined by $h(x) = \arctan(x)$ (principal value of inverse tangent)
   4) i defined by $i(x) = \frac{1}{x^2 - 3x}$

B) The *composition* of two functions $f : A \to B$ and $g : B \to C$ is the function $g \circ f : A \to C$ defined by $(g \circ f)(a) = g(f(a))$ for all $a \in A$.

1) What are formulas that define the compositions $f \circ h$ and $h \circ f$ using the functions from question A?
2) In general, now, if f and g are both injective, prove that $g \circ f$ is injective.
3) If f and g are both surjective, does it follow that $g \circ f$ is surjective? Prove, or give a counterexample.

C) Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$.

1) How many different functions $f : A \to B$ are there? (Explain – what does it mean to say two functions are equal, or different? How are you counting the functions? How could you construct all of them?)
2) How many of the functions from part 1 are *injective*? List the ones that are.
3) How many of the functions from part 1 are *surjective*? Again, list the ones that are.
4) Your answers to parts 2 and 3 of this question should be *the same*. Explain why that is true in intuitive terms (that is, it is not necessary to try to write out a formal proof).

Assignment

Group write-ups due in class Wednesday, September 17.