Mathematics 243, section 1 – Algebraic Structures
Solutions for Exam 3
December 8, 2003

I. A) To show h_a is one-to-one, assume that $h_a(x) = h_a(y)$ for some $x, y \in G$. Then:

$$xa = ya \Rightarrow (xa)a^{-1} = (ya)a^{-1} \Rightarrow x = y$$

Hence h_a is one-to-one. To see that h_a is onto, let $y \in G$ be a general element and consider $x = ya^{-1}$. Then we have

$$h_a(x) = xa = (ya)a^{-1} = y$$

Hence every element of G is in the image of h_a, so h_a is onto.

B) The result from part A says that in the group operation table for G, the column for $a \in G$ contains each element of G exactly once.

II. A) *Comment: Too many people were thinking in a very sloppy way in working on this question. The problem does not ask for the generators of* $\mathbf{Z}_{24}$*, it asks for the generators of the cyclic subgroup* $H = \langle [15] \rangle \subset \mathbf{Z}_{24}$.

Best solution: Recall the general fact in the proofs to know for this exam: In a finite cyclic group $G = \langle a \rangle$ of order n, $\langle a^k \rangle = \langle x^d \rangle$ where $d = \gcd(n, k)$. Here $\mathbf{Z}_{24}$ is cyclic (generator [1] under addition) $n = 24$, and $\gcd(15, 24) = 3$. So the same subgroup is generated by any $[k]$ with $\gcd(k, 24) = 3$. This gives

$$k = [3], [9], [15], \text{ or } [21].$$

Also correct, but more "brute force" approach: Compute the elements of

$$H = \langle [15] \rangle = \{[0], [15], [6], [21], [12], [3], [18], [9]\}.$$

Then check one-by-one to see which of them generate all of H. Only $[3], [9], [15], [21]$ do. For example,

$$\langle [18] \rangle = \{[0], [18], [12], [6]\}$$

only has order 4, so [18] is not a generator of H. But

$$\langle [3] = \{[0], [3], [6], [9], [12], [15], [18], [21]\} = H,$$

so [3] is also one of the generators of H.

B) By direct computation,

$$\sigma = (1532)(24368)(12479) = (14795368)(2) = (14795368)$$

which has order 8, since it is an 8-cycle.

III. A) Since ϕ is a group homomorphism,

$$\phi(e_G) = \phi(e_G \cdot e_G) = \phi(e_G) \cdot \phi(e_G)$$

hence, after multiplying both sides by $\phi(e_G)^{-1}$ in H, we have

$$\begin{aligned} &\phi(e_G)^{-1}\phi(e_G) = \phi(e_G)^{-1}\phi(e_G)\phi(e_G) \\ \Rightarrow\ &e_H = \phi(e_G) \end{aligned}$$

B) Recall the definition:

$$\ker(\phi) = \{x \in G \mid \phi(x) = e_H\}.$$

First, $\ker(\phi)$ is nonempty since by part A, $e_G \in \ker(\phi)$. Next, if $x, y \in \ker(\phi)$, then

$$\phi(xy) = \phi(x)\phi(y) = e_H e_H = e_H$$

Hence $xy \in \ker(\phi)$. Finally, if $x \in \ker(\phi)$, then by a general property of homomorphisms that we proved in class,

$$\phi(x^{-1}) = (\phi(x))^{-1} = e_H^{-1} = e_H$$

Hence $x^{-1} \in \ker(\phi)$. We have shown the three properties needed to see that $\ker(\phi)$ is a subgroup of G.

IV. A) Let $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $B = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$ be *two general matrices in* G – *it is not sufficient to consider only the product* A^2. Then

$$AB = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac - bd & ad + bc \\ -(ad + bc) & ac - bd \end{pmatrix}$$

Since the product AB has the same form (that is, the diagonal entries are equal and the off-diagonal entries are negatives of each other), the product will be in G as long as we know that at least one of these is nonzero. Note that the $\det(A) = a^2 + b^2 \neq 0$ and $\det(B) = c^2 + d^2 \neq 0$. Hence $\det(AB) = \det(A)\det(B) = (a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2 \neq 0$. It follows that at least one of $ac - bd$ and $ad + bc$ is nonzero, and G is closed under matrix products.

B) The general formula for matrix inverses for 2×2 matrices shows if $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, then

$$A^{-1} = \begin{pmatrix} \frac{a}{a^2+b^2} & \frac{-b}{a^2+b^2} \\ \frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{pmatrix}$$

Again, the two diagonal entries are the same, and the off-diagonal entries are negatives. So G is closed under matrix inverses.

Note: It follows from the two parts of this problem that G is a subgroup of $GL_2(\mathbf{R})$, since G is clearly not empty.

Extra Credit. To do this in any reasonable way, you have to remember the 2×2 rotation matrices we saw on one of the problem sets: Let $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. If $\theta = 2\pi/7$, then A will have order 7:

$$A^7 = \begin{pmatrix} \cos(7\theta) & -\sin(7\theta) \\ \sin(7\theta) & \cos(7\theta) \end{pmatrix} = \begin{pmatrix} \cos(2\pi) & -\sin(2\pi) \\ \sin(2\pi) & \cos(2\pi) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$