Mathematics 243, section 1 – Algebraic Structures
Solutions for Exam 1 – October 10, 2003

I.

A) (15) Show that the following proposition is true for *all combinations* of truth values of p, q, r:

$$(p \Rightarrow (q \vee r)) \Leftrightarrow ((p \wedge \sim q) \Rightarrow r)$$

You may use a truth table or any other applicable method.

Solutions: Method 1 (truth table)

p	q	r	$q \vee r$	$p \Rightarrow (q \vee r)$	$(p \wedge \sim q)$	$(p \wedge \sim q) \Rightarrow r$
T	T	T	T	T	F	T
T	T	F	T	T	F	T
T	F	T	T	T	T	T
T	F	F	F	F	T	F
F	T	T	T	T	F	T
F	T	F	T	T	F	T
F	F	T	T	T	F	T
F	F	F	F	T	F	T

The fifth and last columns are the same so the two propositions have the same truth value and $\Leftrightarrow$ will always yield T.

Method 2 (use another logical equivalence $p \Rightarrow q \Leftrightarrow \sim p \vee q$ (twice):

$$\begin{aligned}(p \Rightarrow (q \vee r)) &\Leftrightarrow \sim p \vee (q \vee r) \\ &\Leftrightarrow (\sim p \vee q) \vee r \quad \text{assoc. of } \vee \\ &\Leftrightarrow \sim (p \wedge \sim q) \vee r \quad \text{DeMorgan} \\ &\Leftrightarrow (p \wedge \sim q) \Rightarrow r\end{aligned}$$

B) (15) Let A, B be any two subsets of a universal set U. Show that $(A \cap B)' = A' \cup B'$ (where X' is the complement of X).

Solution: We will show $(A \cap B)' = A' \cup B'$ by showing $(A \cap B)' \subseteq A' \cup B'$ and $A' \cup B' \subseteq (A \cap B)'$. For the first inclusion, let $x \in (A \cap B)'$. Then $x \notin A \cap B$ by the definition of the complement. By the definition of the intersection and DeMorgan's Law for negating an "and" statement, this says $x \notin A$ *or* $x \notin B$. That is, $x \in A'$ or $x \in B'$, so $x \in A' \cup B'$ by the definition of union. This shows $(A \cap B)' \subseteq A' \cup B'$.

For the other inclusion, let $x \in A' \cup B'$. Then $x \in A'$ or $x \in B'$ by the definition of union. As a result, $x \notin A$ or $x \notin B$. Using the same DeMorgan Law as in the previous part, but "in reverse", $x \notin A \cap B$. Hence $x \in (A \cap B)'$. This shows $A' \cup B' \subseteq (A \cap B)'$.

II. Consider the following statement, where A is a nonempty set. "If $f : A \to A$ is a one-to-one mapping, then for all $a \in A$, the set $f^{-1}(\{a\})$ has at most one element."

A) (10) Give the contrapositive form of this statement.

Solution: If for some $a \in A$ the set $f^{-1}(\{a\})$ has more than one element, then f is not a one-to-one mapping.

B) (10) Give the converse of the statement. Is the converse true or false? Explain.

Solution: Converse: If for all $a \in A$ the set $f^{-1}(\{a\})$ has at most one element, then f is a one-to-one mapping. This statement is *true* because it says that if $f(x) = f(y) = a$ (so $x, y \in f^{-1}(\{a\})$), then $x = y$.

III. Let $f, g : \mathbf{Z} \to \mathbf{Z}$ be the mappings

$$f = \begin{cases} 2x+1 & \text{if } x \text{ is even} \\ x-1 & \text{if } x \text{ is odd} \end{cases} \qquad g = \begin{cases} x+1 & \text{if } x \text{ is even} \\ (x-1)/2 & \text{if } x \text{ is odd} \end{cases}$$

A) (10) Let $S = \{1, 2, 3\}$ and $T = \{0, 3, 6\}$. What is $g^{-1}(T) \cap f(S)$?

Solution: $f(S) = \{0, 2, 5\}$. $g^{-1}(T) = \{x | g(x) \in T\} = \{1, 2, 7, 13\}$. So the intersection is $\{2\}$.

B) (10) What is the mapping $f \circ g$?

Solution: To find $f(g(x))$ you must take the parity (i.e. evenness or oddness) of $g(x)$ into account. For instance, if x is even then $g(x) = x + 1$ is odd. Hence we must use that part of the definition of f: If x is even $f(g(x)) = (x+1) - 1 = x$. If x is odd, though, then $g(x)$ can be even or odd. If x is odd and $g(x)$ is odd, then $f(g(x)) = \frac{x-3}{2}$. If x is odd and $g(x)$ is even, then $f(g(x)) = x$.

IV. Let M be the set of all 2×2 matrices with integer entries, and write $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M$. Define a relation R on M by saying $A \; R \; B$ if and only if the matrix product QAQ is equal to B.

A) (5) Show that

$$\begin{pmatrix} 4 & 7 \\ 2 & 3 \end{pmatrix} R \begin{pmatrix} 3 & 2 \\ 7 & 4 \end{pmatrix}$$

is true. (Hint: The matrix product is associative.)

Solution: Multiplying out the matrix product

$$QAQ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 7 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 7 & 4 \end{pmatrix} = B$$

B) (10) For which matrices A is ARA true?

Solution: The general pattern you can see from the above computation is that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$QAQ = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

This equals A again only when $a = d$ and $b = c$. The general matrix A for which ARA is true looks like:

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

where a, b are any integers.

V. (15) Prove by mathematical induction: If $n \in \mathbf{Z}^+$, then

$$a + (a+d) + (a+2d) + \cdots + (a+(n-1)d) = \frac{n}{2}(2a + (n-1)d).$$

Solution: The base case is $n = 1$, where the left hand side is just one term a, and the right hand side is $\frac{1}{2}(2a + 0d) = a$. Hence the base case is true.

Now, assume we know the formula for $n = k$:

$$a + (a+d) + (a+2d) + \cdots + (a+(k-1)d) = \frac{k}{2}(2a + (k-1)d),$$

consider the sum on the left for $n = k+1$, and use the induction hypothesis on the first part of the sum:

$$\begin{aligned} a + (a+d) + (a+2d) + \cdots + (a+(k-1)d) + (a+kd) &= \frac{k}{2}(2a + (k-1)d) + (a+kd) \\ &= ka + \frac{k(k-1)}{2}d + a + kd \\ &= (k+1)a + \frac{k^2 - k + 2k}{2}d \\ &= (k+1)a + \frac{k(k+1)}{2}d \\ &= \frac{(k+1)}{2}(2a + kd) \end{aligned}$$

which is what we wanted to show.