Mathematics 136 – AP Calculus
Summary of Trigonometric Substitutions
Fall 2009

The trigonometric substitution method handles many integrals containing expressions like

$$\sqrt{a^2 - x^2}, \sqrt{x^2 + a^2}, \sqrt{x^2 - a^2}$$

(possibly including expressions without the square roots!) The basis for this approach is the trigonometric identities

$$\begin{aligned} 1 &= \sin^2\theta + \cos^2\theta \\ \Rightarrow \sec^2\theta &= \tan^2\theta + 1. \end{aligned}$$

from which we derive other related identities:

$$\begin{aligned} \sqrt{a^2 - (a\sin\theta)^2} &= a\cos\theta \\ \sqrt{(a\tan\theta)^2 + a^2} &= a\sec\theta \\ \sqrt{(a\sec\theta)^2 - a^2} &= a\tan\theta \end{aligned}$$

Hence,

1. If our integral contains $\sqrt{a^2 - x^2}$, the substitution $x = a\sin\theta$ will convert this radical to the simpler form $a\cos\theta$.
2. If our integral contains $\sqrt{x^2 + a^2}$, the substitution $x = a\tan\theta$ will convert this radical to the simpler form $a\sec\theta$.
3. If our integral contains $\sqrt{x^2 - a^2}$, the substitution $x = a\sec\theta$ will convert this radical to the simpler form $a\tan\theta$.

Then we substitute for the rest of the integral, integrate the resulting trigonometric form, and convert back to the original variable.

Two Examples

A) Compute $\int \frac{u^2}{\sqrt{a^2-u^2}}\ du$. Solution: The $\sqrt{a^2 - u^2}$ tells us that we want the sine substitution: $u = a\sin\theta$. Then $du = a\cos\theta\ d\theta$, and the integral becomes:

$$\int \frac{a^3\sin^2\theta\cos\theta\ d\theta}{a\cos\theta} = a^2\int \sin^2\theta\ d\theta$$

We apply the reduction formula for powers of sin (this is # 73 in Stewart's table of integrals from the Reference Pages handout from last week):

$$a^2\int \sin^2\theta\ d\theta = \frac{-a^2}{2}\sin\theta\cos\theta + \frac{a^2}{2}\int\ d\theta = \frac{-a^2}{2}\sin\theta\cos\theta + \frac{a^2}{2}\theta + C.$$

Then, we convert back to functions of u using the substitution equation $u = a\sin\theta$. From this,

$$\theta = \arcsin(u/a), \qquad \cos\theta = \sqrt{a^2-u^2}/a, \qquad \sin\theta = u/a.$$

So the integral is

$$\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = \frac{-1}{2}u\sqrt{a^2-u^2} + \frac{a^2}{2}\arcsin(u/a) + C.$$

Note that the final answer here matches # 34 in the table of integrals.

B) Compute $\int \frac{dx}{x\sqrt{x^2+16}}$. Solution: The $\sqrt{x^2+16}$ indicates that we want the tangent substitution $x = 4\tan\theta$. Then $dx = 4\sec^2\theta\ d\theta$ and the integral becomes:

$$\int \frac{4\sec^2\theta\ d\theta}{4\tan\theta\cdot 4\sec\theta} = \frac{1}{4}\int \frac{\sec\theta}{\tan\theta}\,d\theta = \frac{1}{4}\int \frac{1}{\sin\theta}\,d\theta$$

Since $1/\sin(\theta) = \csc(\theta)$, from # 15 in Stewart's table of integrals,

$$\frac{1}{4}\int \csc(\theta)\ d\theta = \frac{1}{4}\ln|\csc(\theta) - \cot(\theta)| + C$$

Then, from $x = 4\tan\theta$, we get $\cos\theta = \frac{4}{\sqrt{x^2+16}}$ and $\sin\theta = \frac{x}{\sqrt{x^2+16}}$. Hence

$$\csc(\theta) = \frac{\sqrt{x^2+16}}{x} \quad \text{and} \quad \cot(\theta) = \frac{4}{x}.$$

The integral equals:

$$\frac{1}{4}\ln\left|\frac{\sqrt{x^2+16}}{x} - \frac{4}{x}\right| + C.$$