The trigonometric substitution method handles many integrals containing expressions like
\[ \sqrt{a^2 - x^2}, \sqrt{x^2 + a^2}, \sqrt{x^2 - a^2} \]
(possibly including expressions without the square roots!) The basis for this approach is the trigonometric identities
\[
1 = \sin^2 \theta + \cos^2 \theta \\
\Rightarrow \sec^2 \theta = \tan^2 \theta + 1.
\]
from which we derive other related identities:
\[
\sqrt{a^2 - (a \sin \theta)^2} = a \cos \theta \\
\sqrt{(a \tan \theta)^2 + a^2} = a \sec \theta \\
\sqrt{(a \sec \theta)^2 - a^2} = a \tan \theta
\]
Hence,
1. If our integral contains \( \sqrt{a^2 - x^2} \), the substitution \( x = a \sin \theta \) will convert this radical to the simpler form \( a \cos \theta \).
2. If our integral contains \( \sqrt{x^2 + a^2} \), the substitution \( x = a \tan \theta \) will convert this radical to the simpler form \( a \sec \theta \).
3. If our integral contains \( \sqrt{x^2 - a^2} \), the substitution \( x = a \sec \theta \) will convert this radical to the simpler form \( a \tan \theta \).

Then we substitute for the rest of the integral, integrate the resulting trigonometric form, and convert back to the original variable.

Two Examples

A) Compute \( \int \frac{u^2}{\sqrt{a^2 - u^2}} \, du \). Solution: The \( \sqrt{a^2 - u^2} \) tells us that we want the sine substitution: \( u = a \sin \theta \). Then \( du = a \cos \theta \, d\theta \), and the integral becomes:
\[
\int \frac{a^3 \sin^2 \theta \cos \theta \, d\theta}{a \cos \theta} = a^2 \int \sin^2 \theta \, d\theta
\]
We apply the reduction formula for powers of sin (this is \# 73 in Stewart’s table of integrals from the Reference Pages handout from last week):
\[
a^2 \int \sin^2 \theta \, d\theta = \frac{-a^2}{2} \sin \theta \cos \theta + \frac{a^2}{2} \int \, d\theta = \frac{-a^2}{2} \sin \theta \cos \theta + \frac{a^2}{2} \theta + C.
\]
Then, we convert back to functions of $u$ using the substitution equation $u = a \sin \theta$. From this,

$$
\theta = \arcsin(u/a), \quad \cos \theta = \sqrt{a^2 - u^2}/a, \quad \sin \theta = u/a.
$$

So the integral is

$$\int \frac{u^2}{\sqrt{a^2 - u^2}} \, du = -\frac{1}{2} u \sqrt{a^2 - u^2} + \frac{a^2}{2} \arcsin(u/a) + C.$$

Note that the final answer here matches # 34 in the table of integrals.

B) Compute $\int \frac{dx}{x \sqrt{x^2 + 16}}$. Solution: The $\sqrt{x^2 + 16}$ indicates that we want the tangent substitution $x = 4 \tan \theta$. Then $dx = 4 \sec^2 \theta \, d\theta$ and the integral becomes:

$$\int \frac{4 \sec^2 \theta \, d\theta}{4 \tan \theta \cdot 4 \sec \theta} = \frac{1}{4} \int \frac{\sec \theta}{\tan \theta} \, d\theta = \frac{1}{4} \int \frac{1}{\sin \theta} \, d\theta$$

Since $1/ \sin(\theta) = \csc(\theta)$, from # 15 in Stewart’s table of integrals,

$$\frac{1}{4} \int \csc(\theta) \, d\theta = \frac{1}{4} \ln |\csc(\theta) - \cot(\theta)| + C$$

Then, from $x = 4 \tan \theta$, we get $\cos \theta = \frac{4}{\sqrt{x^2 + 16}}$ and $\sin \theta = \frac{x}{\sqrt{x^2 + 16}}$. Hence

$$\csc(\theta) = \frac{\sqrt{x^2 + 16}}{x} \quad \text{and} \quad \cot(\theta) = \frac{4}{x}.$$

The integral equals:

$$\frac{1}{4} \ln \left| \frac{\sqrt{x^2 + 16}}{x} - \frac{4}{x} \right| + C.$$