Mathematics 136 – AP Calculus
Integration by Partial Fractions
Fall 2009

The *partial fractions* method applies to (in principle, *all*) rational functions

$$h(x) = \frac{f(x)}{g(x)} = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$$

(or to functions that can be brought to this form by a preliminary substitution). The steps involved are:

1. If $n \geq m$, first *divide* $g(x)$ into $f(x)$ using polynomial division to write $f(x) = q(x)g(x) + r(x)$, where the degree of $r(x)$ is less than m. This yields

   $$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

   and

   $$\int \frac{f(x)}{g(x)}\,dx = \int q(x)\,dx + \int \frac{r(x)}{g(x)}\,dx$$

2. Now, assuming we have a rational function where the degree of the numerator is strictly less than the degree of the denominator, *factor* the denominator completely into linear and quadratic factors. The quadratic factors will have no real roots when they cannot be factored further. (It is a theorem that any polynomial with real coefficients can be factored in this way. The "bottleneck" in applying this method in very hard examples, though, is often finding the appropriate factorization. *Needless to say, we will not be doing any of those very hard examples(!)*
3. Set up the partial fractions.

   a) If $(x+a)^e$ is the highest power of a linear polynomial that divides $g(x)$, then the partial fractions will include a group of terms

   $$\frac{A_1}{x+a} + \frac{A_2}{(x+a)^2} + \cdots + \frac{A_e}{(x+a)^e}$$

   b) If $(ax^2 + bx + c)$ is a quadratic with no real roots that divides $g(x)$, then the partial fractions will include a term

   $$\frac{Bx + C}{ax^2 + bx + c}.$$

   Note the *linear polynomial on the top.* The reason you need both the Bx and the C will become apparent by doing some examples. It is not possible to "match" every possible numerator unless both the Bx and the C are included. *Note:* We will only be doing examples where quadratic factors appear to the first power in the factorization of the denominator. There is a more general form of partial fractions that applies if you have something like $(x^2+6x+10)^3$ in the factorization, though. You need a sum of partial fractions in that case, something like in step 3 above.

4. Solve for the coefficients in the partial fractions. This can be done by clearing denominators, then substituting in x-values and/or equating coefficients on both sides of the resulting equation.
5. Integrate the partial fractions.

Worked Example

Here is an example illustrating many of these steps. Suppose we need to integrate

$$\int \frac{x^5+4x+1}{x^4+9x^2}\,dx.$$

The degree of the top is larger, so we divide first:

$$x^5+4x+1 = x(x^4+9x^2) + (-9x^3+4x+1)$$

so

$$\frac{x^5+4x+1}{x^4+9x^2} = x + \frac{-9x^3+4x+1}{x^4+9x^2}$$

The denominator factors as $x^4+9x^2 = x^2(x^2+9)$. So the partial fractions are

$$\frac{-9x^3+4x+1}{x^4+9x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+9}.$$

Clearing denominators,

$$-9x^3+4x+1 = Ax(x^2+9) + B(x^2+9) + (Cx+D)x^2.$$

Substituting $x=0$ we see $1 = 9B$ so $B = 1/9$. From the coefficient of x we see $4 = 9A$, so $A = 4/9$. Then from the coefficients of x^3, $A+C = -9$, so $C = -9-4/9 = -85/9$ and finally from the coefficient of x^2, $0 = B+D$, so $D = -1/9$. This gives

$$\begin{aligned}\int \frac{x^5+4x+1}{x^4+9x^2}\,dx &= \int x + \frac{-9x^3+4x+1}{x^4+9x^2}\,dx \\ &= \int x + \frac{4/9}{x} + \frac{1/9}{x^2} + \frac{(-85/9)x+(-1/9)}{x^2+9}\,dx \\ &= \frac{x^2}{2} + \frac{4}{9}\ln|x| - \frac{1}{9}\frac{1}{x} - \frac{85}{18}\ln(x^2+9) - \frac{1}{27}\tan^{-1}(x/3) + C.\end{aligned}$$

A final comment – note that if the Cx term were not present, then there would be no way to "match" the numerator $-9x^3+4x+1$ when we add up the partial fractions.