Mathematics 136 – Advanced Placement Calculus
Solutions to Ungraded Problems on Problem Set 3
September 29, 2009

2.8/10

(a) The rate of increase of the population is small at first, then much larger around $t = 8$, then smaller again (tending to zero, apparently) as t increases farther.

(b) The rate of increase is largest at $t = 8$.

(c) The graph of the population function is concave up for $0 < t < 8$ and concave down for $8 < t < 18$.

(d) At $t = 8$, the population is 350. The inflection point on the graph is at $(8, 350)$.

2.8/16.

(a) f is increasing where f' is positive, so on $(1, 6)$ and for $x > 8$. f is decreasing where f' is negative, so on $(0, 1)$ and $(6, 8)$.

(b) f has local minima at $x = 1$ and $x = 8$ and a local maximum at $x = 6$ (using what we stated as the First Derivative Test in class on 9/29).

(c) f is concave up where f' is increasing, so on $(0, 2), (3, 5), (7, \infty)$. f is concave down where f' is decreasing, so on $(2, 3)$ and $(5, 7)$.

(d) and (e) The graph should follow parts (a), (b), (c). Note that f is increasing all the way from $x = 1$ to $x = 6$; what changes on that interval is the concavity (three changes!). There are inflection points at $x = 2, 3, 5$ and $x = 7$.

3.1/7. $f'(x) = 3x^2 - 4$.

3.1/25. $x = Ay^{-10} + Be^y$ (where A, B are constants). So $\frac{dx}{dy} = -10Ay^{-11} + Be^y$.

3.2/11. By the quotient rule:

$$y' = \frac{(1 - x^2)(3x^2) - x^3 \cdot (-2x)}{(1 - x^2)^2} = \frac{3x^2 - x^4}{(1 - x^2)^2}.$$

3.2/22. By the quotient rule:

$$f'(x) = \frac{(x + e^x)(-xe^x - e^x) - (1 - xe^x)(1 + e^x)}{(x + e^x)^2} = \frac{-x^2e^x - e^{2x} - e^x - 1}{(x + e^x)^2} \text{ after simplifying.}$$

3.2/45. The idea of this problem is to use the product or quotient rule as appropriate, and read the values of f, g, and the derivatives of f, g from the graph, recalling that $f'(a), g'(a)$ are slopes of tangent lines.

(a)

$$u'(1) = f(1)g'(1) + f'(1)g(1) = 2 \cdot (-1) + 2 \cdot 1 = 0.$$

(b)

$$v'(5) = \frac{g(5)f'(5) - f(5)g'(5)}{(g(5))^2} = \frac{2 \cdot (-1/3) - 3 \cdot (2/3)}{2^2} = -\frac{2}{3}.$$

3.3/5. By the product rule, $y' = \sec\theta \cdot \sec^2\theta + \tan\theta \cdot \sec\theta\tan\theta = \sec\theta(\sec^2\theta + \tan^2\theta)$. This can also be rewritten in various ways by using trig identities, recalling that $1 + \tan^2\theta = \sec^2\theta$.

3.3/35.

(a) Since $x(t) = 8\sin(t)$, $v(t) = x'(t) = 8\cos(t)$ and $a(t) = x''(t) = -8\sin(t)$.

(b) At $t = \frac{2\pi}{3}$, $v = 8\cos\left(\frac{2\pi}{3}\right) = -4$ and $a = -8\sin\left(\frac{2\pi}{3}\right) = -4\sqrt{3}$. This means that the mass is moving to the left ($v < 0$).

3.4/15. The general derivative rule for the exponential functions is

$$\frac{d}{dx}a^x = a^x \ln(a).$$

(This can be seen by rewriting $a^x = e^{x\ln(a)}$ and using the chain rule.) So $h'(t) = 3t^2 - 3^t\ln(3)$.

3.4/30. Think of the function $f(t)$ in the form $f(t) = \left(\frac{t}{t^2+4}\right)^{1/2}$ and use the chain rule and quotient rule:

$$f'(t) = \frac{1}{2}\left(\frac{t}{t^2+4}\right)^{-1/2}\left(\frac{(t^2+4) - t \cdot (2t)}{(t^2+4)^2}\right).$$

This can be simplied to:

$$f'(t) = \frac{4 - t^2}{2t^{1/2}(t^2+4)^{3/2}}.$$

3.5/15. By implicit differentiation:

$$-e^y\sin(x) + \cos(x) \cdot e^y y' = \cos(xy) \cdot (y + xy').$$

Solving for y':

$$(\cos(x)e^y - x\cos(xy))y' = y\cos(xy) + e^y\sin(x).$$

So

$$y' = \frac{y\cos(xy) + e^y\sin(x)}{\cos(x)e^y - x\cos(xy)}.$$