I. Consider the function \( f(x) = x^3 + 6x^2 - 15x \).

A) (10) Find all the critical numbers of \( f \) and classify each as a local maximum, local minimum, or neither.

_Solution:_ We have \( f'(x) = 3x^2 + 12x - 15 = 3(x + 5)(x - 1) \). This is defined for all \( x \) and zero only at \( x = -5, 1 \) so those are the critical numbers. We have \( f''(x) = 6x + 12 \). Hence \( f''(-5) = -18 \) and \( f''(1) = 18 \). By the Second Derivative Test, \( x = -5 \) is a local maximum, and \( x = 1 \) is a local minimum. (Note: this can also be determined by the First Derivative Test, finding the intervals where \( f' \) is positive (\( x < -5 \) and \( x > 1 \)) and the interval where \( f' \) is negative (\( -5 < x < 1 \)).

B) (5) What are the absolute (global) maximum and minimum values of \( f(x) \) on the interval \([0, 3]\)?

_Solution:_ The critical number \( x = 1 \) is in this interval. \( f(0) = 0 \), \( f(1) = -8 \) and \( f(3) = 36 \). So the absolute maximum is \( f(3) = 36 \) and the absolute minimum is \( f(1) = -8 \).

II. (10) Evaluate the following limit:

\[
\lim_{\theta \to 0} \frac{\cos(5\theta) - 1}{\theta^2}.
\]

_Solution:_ This is indeterminate of the form \( 0/0 \), so we can apply L’Hopital’s Rule (twice):

\[
\lim_{\theta \to 0} \frac{\cos(5\theta) - 1}{\theta^2} = \lim_{\theta \to 0} \frac{-5 \sin(5\theta)}{2\theta} \quad (\text{still } 0/0)
\]

\[
= \lim_{\theta \to 0} \frac{-25 \cos(5\theta)}{2} = \frac{-25}{2}.
\]

III. The velocity of a accelerating car is measured at each of the following times, yielding a table of values:

<table>
<thead>
<tr>
<th>( t ) (sec)</th>
<th>( v(t) ) (ft/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.35</td>
</tr>
<tr>
<td>3</td>
<td>5.12</td>
</tr>
<tr>
<td>5</td>
<td>7.86</td>
</tr>
<tr>
<td>7</td>
<td>9.07</td>
</tr>
<tr>
<td>9</td>
<td>11.20</td>
</tr>
</tbody>
</table>

A) (5) Give an estimate for the total distance traveled by the car between \( t = 1 \) and \( t = 11 \) using a left-hand Riemann sum for the velocity function and \( \Delta t = 2 \).

_Solution:_ The left-hand Riemann sum estimate is

\[
v(1)\Delta t + \Delta t + v(5)\Delta t + v(9)\Delta t = 2 + 7 + 10.2 + 15.6 + 18 = 52.8 \text{ ft}.
\]

B) (5) Is your result from A) less than or greater than the actual total distance traveled, assuming \( v \) is always increasing between \( t = 1 \) and \( t = 11 \)? Explain.

_Solution:_ If \( v \) is always increasing on this interval, then each term in the left-hand Riemann sum _underestimates_ the actual distance traveled on that interval. Hence the answer in A is _less_ than the actual distance traveled.
IV. Compute each of the following integrals. Show all work.

A) \(\int_{1}^{4} x^2 - 3\sqrt{x} + 4 \, dx\)

Solution: By the Evaluation Theorem, this gives:
\[
\frac{x^3}{3} - 2x^{3/2} + 4x \bigg|_{1}^{4} = \frac{64}{3} - 16 + \frac{1}{3} + 2 - 4 = 19.
\]

B) \(\int \sec^2(2x) \frac{3}{\sqrt{\tan(2x)} + 4} \, dx\)

Solution: Let \(u = \tan(2x) + 4\). Then \(du = 2\sec^2(2x) \, dx\), and the form is
\[
\frac{1}{2} \int u^{1/3} \, du = \frac{1}{2} \cdot \frac{3}{4} u^{4/3} + C = \frac{3}{8} (\tan(2x) + 4)^{4/3} + C.
\]

Integrate using the indicated method and the table (say which entry you are using if you use one).

C) \(\int x^2 e^{4x} \, dx\) (full credit only for showing all work; you can check your result by the table)

Solution: Integrate by parts twice making \(u\) the power of \(x\) both times:
\[
\int x^2 e^{4x} \, dx = \frac{x^2 e^{4x}}{4} - \frac{2}{4} \int xe^{4x} \, dx
= \frac{x^2 e^{4x}}{4} - \frac{1}{2} \left( \frac{xe^{4x}}{4} - \frac{1}{4} \int e^{4x} \, dx \right)
= \frac{x^2 e^{4x}}{4} - \frac{xe^{4x}}{8} + \frac{e^{4x}}{32} + C.
\]

D) \(\int \frac{\sqrt{25 - x^2}}{x} \, dx\)

Solution: From the form of the integrand, let \(x = 5 \sin \theta\), so \(dx = 5 \cos \theta \, d\theta\). Then the integral can be evaluated by using the trigonometric identity \(\cos^2 \theta = 1 - \sin^2 \theta\):
\[
\int \frac{\sqrt{25 - x^2}}{x} \, dx = \int \frac{5 \cos \theta}{5 \sin \theta} \cdot 5 \cos \theta \, d\theta
= 5 \int \frac{\cos^2 \theta}{\sin \theta} \, d\theta
= 5 \int \frac{1 - \sin^2 \theta}{\sin \theta} \, d\theta
= 5 \int \csc \theta - \sin \theta \, d\theta
= 5 (\ln |\csc \theta - \cot \theta| + \cos \theta) + C \quad \# \ 15 \text{ in table}
= 5 \ln \left| \frac{5}{x} - \frac{\sqrt{25 - x^2}}{x} \right| + \sqrt{25 - x^2} + C.
\]
E) (12.5) Use partial fractions:

\[ \int \frac{x + 4}{(x + 1)(x^2 + 16)} \, dx. \]

Solution: From the form of the denominator we set up the partial fractions:

\[
\frac{2x + 19}{(x + 1)(x^2 + 16)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 16}
\]

\[2x + 19 = A(x^2 + 16) + (Bx + C)(x + 1).\]

Equating coefficients gives \(A + B = 0\), \(B + C = 2\) and \(16A + C = 19\). So \(A = 1\), \(B = -1\), and \(C = 3\). Integrating with #15 in table, we obtain:

\[\int \frac{dx}{x + 1} + \int \frac{-x + 3}{x^2 + 16} \, dx = \ln|x + 1| - \frac{1}{2} \ln|x^2 + 16| + \frac{3}{4} \tan^{-1}(x/4) + C.\]

V. A) (15) A fish swimming against a current of \(u\) ft/sec (constant) for a distance \(L\) (constant) will expend energy

\[E(v) = \frac{Lv^3}{v - u}\]

if it swims at \(v\) ft/sec \((u < v)\). What \(v\) minimizes the total energy expended?

Solution: By the quotient rule,

\[E'(v) = \frac{(v - u) \cdot 3Lv^2 - Lv^3 \cdot 1}{(v - u)^2},\]

and \(E'(v) = 0\) when the numerator is zero, so \(3L(v - u)v^2 = Lv^3\). Dividing through by \(Lv^2\), \(3(v - u) = v\), so \(v = \frac{3u}{4}\). This a minimum by the first derivative test.

B) (15) Using the definition (not the Evaluation Theorem) compute

\[\int_{2}^{3} x^2 + 4x \, dx.\]

The following formulas may be useful:

\[\sum_{i=1}^{n} i = \frac{n(n + 1)}{2} \quad \sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}.\]

Solution: For the right-hand Riemann sum, we have \(\Delta x = \frac{3 - 2}{n} = \frac{1}{n}\) and \(x_i = 2 + \frac{i}{n}\). Then

\[\int_{2}^{3} x^2 + 4x \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left( \left( 2 + \frac{i}{n} \right)^2 + 4 \left( 2 + \frac{i}{n} \right) \right) \frac{1}{n}
\]

\[= \lim_{n \to \infty} \sum_{i=1}^{n} \left( \frac{i^2}{n^2} + \frac{8i}{n} + 12 \right) \frac{1}{n}
\]

\[= \lim_{n \to \infty} \left( \frac{1}{n^3} \sum_{i=1}^{n} i^2 + \frac{8}{n^2} \sum_{i=1}^{n} i + 12 \right)
\]

\[= \lim_{n \to \infty} \frac{n(n + 1)(2n + 1)}{n^3} + \frac{8n(n + 1)}{2n^2} + 12
\]

\[= \frac{1}{3} + \frac{8}{2} + 12 = \frac{49}{3}.
\]
(Check by Evaluation Theorem:

\[ \int_{2}^{3} x^2 + 4x \, dx = \frac{x^3}{3} + 2x^2 \bigg|_{2}^{3} = 9 + 18 - \frac{8}{3} - 8 = \frac{49}{3}. \] )