Mathematics 136 – Advanced Placement Calculus
Exam 3 – Review Sheet – Solutions for Practice Problems
November 24, 2009

Sample Exam Questions

This list is *much longer* than the actual exam will be (to give you some idea of the range of different questions that might be asked). Unless otherwise directed, you may use any entry of the Table of Integrals from the text that applies.

I. For each of the following integrals, say why the integral is improper, determine if the integral converges, and if so, find its value.

A) $\int_1^\infty \frac{1}{\sqrt[5]{x}}\, dx$

Solution: This is improper because of the infinite interval of integration. To see whether it converges, we need to study the following limit:

$$\begin{aligned}\int_1^\infty \frac{1}{\sqrt[5]{x}}\, dx &= \lim_{b\to\infty} \int_1^b x^{-1/5}\, dx\\ &= \lim_{b\to\infty} \frac{5}{4} x^{4/5}\Big|_1^b\\ &= \lim_{b\to\infty} \frac{5b^{4/5}-5}{4}\\ &= +\infty\end{aligned}$$

Since this is not finite, the integral *diverges.*

B) $\int_0^2 \frac{dx}{x^2-7x+6}$

Solution: The integral is improper since the function $\frac{1}{x^2-7x+6} = \frac{1}{(x-6)(x-1)}$ is discontinuous at $x = 1$ in the interval of integration. The integral converges only if

$$\lim_{b\to 1^-} \int_0^b \frac{dx}{x^2-7x+6}$$

and

$$\lim_{a\to 1^+} \int_a^2 \frac{dx}{x^2-7x+6}$$

both exist. By partial fractions, we have

$$\frac{1}{x^2-7x+6} = \frac{\frac{1}{5}}{x-6} + \frac{\frac{-1}{5}}{x-1}.$$

So

$$\lim_{b\to 1^-}\int_0^b \frac{dx}{x^2-7x+6} = \lim_{b\to 1^-}\int_0^b \frac{\frac{1}{5}}{x-6}+\frac{\frac{-1}{5}}{x-1}\,dx$$
$$= \lim_{b\to 1^-} \frac{1}{5}\ln|x-6| - \frac{1}{5}\ln|x-1|\Big|_0^b$$
$$= \lim_{b\to 1^-} \frac{1}{5}\left(\ln|b-6| - \ln|b-1| - \ln(6)\right).$$

This limit is not finite since $\lim_{b\to 1^-}\ln|b-1| = -\infty$. Hence this integral *diverges*.

C) $\int_0^\infty xe^{-3x}\,dx$

Solution: This is improper because of the infinite interval of integration. Integrating by parts, or using the table, we have:

$$\lim_{b\to\infty}\int_0^b xe^{-3x}\,dx = \lim_{b\to\infty} -\frac{xe^{-3x}}{3} - \frac{e^{-3x}}{9}\Big|_0^b$$
$$= \lim_{b\to\infty} -\frac{b}{3e^{3b}} - \frac{1}{9e^{3b}} + 0 + \frac{1}{9}$$
$$= \frac{1}{9}$$

(the limit of the first term is zero, for instance, by L'Hopital's Rule). So the integral *converges*.

D) For which values of a is $\int_0^\infty e^{ax}\sin(x)\,dx$ convergent? Evaluate the integral for those a.

Solution: This is improper because of the infinite interval of integration. By # 98 in the table of integrals (or parts),

$$\lim_{b\to\infty}\int_0^b e^{ax}\sin(x)\,dx = \lim_{b\to\infty}\frac{e^{ax}}{a^2+1}(a\sin x - \cos x)\Big|_0^b$$
$$= \lim_{b\to\infty}\frac{e^{ab}}{a^2+1}(a\sin b - \cos b) + \frac{1}{a^2+1}.$$

This limit exists and is finite only if $a < 0$. If so, the limit is $\frac{1}{a^2+1}$.

II.

(A) Let R be the region in the plane bounded by $y = 3 - x^2$ and the x-axis.

(1) Sketch the region R.

Sketch omitted – $y = 3 - x^2$ is a parabola opening down, intersecting the x-axis at $x = \pm\sqrt{3}$.

(2) Find the area of R.

Solution:

$$A = \int_{-\sqrt{3}}^{\sqrt{3}} 3 - x^2 \; dx = 3x - \frac{x^3}{3}\Big|_{-\sqrt{3}}^{\sqrt{3}} = 4\sqrt{3}.$$

(3) Find the volume of the solid generated by rotating R about the x-axis.

Solution:

$$V = \int_{-\sqrt{3}}^{\sqrt{3}} \pi(3 - x^2)^2 \; dx = \frac{48\pi\sqrt{3}}{5}.$$

(4) Find the coordinates $\overline{x}, \overline{y}$ of its centroid.

Solution: The coordinates of the centroid are:

$$\overline{x} = \frac{1}{A}\int_{-\sqrt{3}}^{\sqrt{3}} x(3 - x^2) \; dx = 0.$$

(This can also be seen intuitively by the symmetry of the region about the y-axis.) Then

$$\overline{y} = \frac{1}{A}\int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{2}(3 - x)^2 \; dx.$$

Reusing the computations from part 3, we see that this is

$$\overline{y} = \frac{1}{2} \cdot \frac{1}{4\sqrt{3}} \cdot \frac{48\sqrt{3}}{5} = \frac{6}{5}.$$

(B) Let R be the region in the plane bounded by $y = \cos(\pi x)$, $x = 0$, $x = 1/2$, and $y = 0$.

(1) Sketch the region R.

Sketch omitted. This is the part of a cosine graph from a maximum at $y = 1$ down to the next zero.

(2) Find the area of R.

Solution: The area is

$$A = \int_0^{1/2} \cos(\pi x) \; dx = \frac{1}{\pi}\sin(\pi x)\Big|_0^{1/2} = \frac{1}{\pi}.$$

(3) Find the volume of the solid generated by rotating R about the x-axis.

Solution: Using # 74 in the table after a preliminary substitution $u = \pi x$, the volume is:

$$V = \int_0^{1/2} \pi\cos^2(\pi x) \; dx = \frac{1}{2}\cos(\pi x)\sin(\pi x) + \frac{\pi x}{2}\Big|_0^{1/2} = \frac{\pi}{4}.$$

(4) Find the coordinates $\overline{x}, \overline{y}$ of its centroid.

Solution: The coordinates of the centroid are

$$\overline{x} = \frac{1}{A}\int_0^{1/2} x\cos(\pi x)\ dx = \pi \cdot \left(\frac{x\sin(\pi x)}{\pi} + \frac{\cos(\pi x)}{\pi^2}\Bigg|_0^{1/2} \right) = \frac{\pi - 2}{2\pi} \doteq .182$$

and (reusing part 3):

$$\overline{y} = \frac{1}{A}\int_0^{1/2} \frac{1}{2}\cos^2(\pi x)\ dx = \frac{\pi}{8}.$$

III. The height of a monument is 20m. The horizontal cross-section of the monument at x meters from the top is an isosceles right triangle with legs $x/4$ meters. Find the volume of the monument.

Solution: The volume is the integral of the area of the cross-section. Since the cross-section is a right triangle, the area is $A(x) = \frac{1}{2} \cdot \frac{x}{4} \cdot \frac{x}{4} = \frac{x^2}{32}$. Then

$$\begin{aligned} V &= \int_0^{20} \frac{x^2}{32}\ dx \\ &= \frac{x^3}{96}\Bigg|_0^{20} \\ &= \frac{8000}{96} = \frac{250}{3}. \end{aligned}$$

cubic meters.

IV.

(A) Set up and evaluate the integral to compute the arclength of the curve $y = 3x^2$, $0 \leq x \leq 2$.

Solution: We have $y' = 6x$, so by # 21 in the table with $u = 6x$ and $a = 1$,

$$\begin{aligned} L &= \int_0^2 \sqrt{1 + (y')^2}\ dx \\ &= \int_0^2 \sqrt{1 + 36x^2}\ dx \\ &= \frac{1}{6}\left(3x\sqrt{1 + 36x^2} + \frac{1}{2}\ln(6x + \sqrt{1 + 36x^2}) \right)\Bigg|_0^2 \\ &= \sqrt{145} + \frac{1}{12}\ln(12 + \sqrt{145}) \\ &\doteq 12.3 \end{aligned}$$

(B) Set up and evaluate the integral to compute the arclength of the curve $y = \frac{1}{6}(x^2+4)^{3/2}$, $0 \leq x \leq 3$. (Hint: the arclength integral simplifies to a manageable form if you are careful with the algebra.)

Solution: We have $y' = \frac{x}{2}(x^2+4)^{1/2}$. So

$$\begin{aligned}
L &= \int_0^3 \sqrt{1+(y')^2}\ dx \\
&= \int_0^3 \sqrt{1+\frac{x^2}{4}\cdot(x^2+4)}\ dx \\
&= \int_0^3 \frac{1}{2}\sqrt{x^4+4x^2+4}\ dx \\
&= \int_0^3 \frac{1}{2}(x^2+2)\ dx \\
&= \frac{x^3}{6}+x\Big|_0^3 \\
&= \frac{15}{2}.
\end{aligned}$$

V.

(A) Find the average value of $f(x)=\sqrt{1-x^2}$ on the interval $[0,1/2]$. (Use trigonometric substitution, not the table.)

Solution: The average value is:

$$\begin{aligned}
f_{ave} &= \frac{1}{1/2-0}\int_0^{1/2}\sqrt{1-x^2}\ dx \\
&= 2\int_0^{\pi/6}\sqrt{1-\sin^2\theta}\ \cos\theta\ d\theta \qquad \text{(letting } x=\sin\theta\text{)} \\
&= 2\int_0^{\pi/6}\cos^2\theta\ d\theta \\
&= \sin\theta\cos\theta+\theta|_0^{\pi/6} \\
&= \frac{\sqrt{3}}{4}+\frac{\pi}{6} \\
&\doteq .957
\end{aligned}$$

(B) Find the average value of $f(x)=x\sqrt{1+x^4}$ on the interval $[0,2]$.

Solution: We have

$$\begin{aligned}
f_{ave} &= \frac{1}{2-0}\int_0^2 x\sqrt{1+x^4}\ dx \\
&= \frac{1}{4}\int_0^4 \sqrt{1+u^2}\ du \qquad (\text{letting } u = x^2,\ du = 2x\ dx) \\
&= \frac{1}{4}\left(\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln(u+\sqrt{1+u^2})\bigg|_0^4\right) \qquad \text{by \# 21 in the table} \\
&= \frac{\sqrt{17}}{2} + \frac{1}{8}\ln(4+\sqrt{17}) \\
&\doteq 2.32
\end{aligned}$$

VI. A cylindrical tank 10 meters tall and with base radius 3 meters is full of water (density 1000kg per cubic meter). Find the work done in pumping all the water out the top of the tank.

Solution: Consider the slice y meters above the bottom of the tank (a cylinder of radius 3 and height Δy). The work done lifting that slice is the mass of the slice, times $g = 9.8$ meters per second squared, times the distance to the top, which is $10 - y$:

$$W_{slice} = \pi(3)^2\Delta y \cdot 1000 \cdot 9.8 \cdot (10-y).$$

Hence the total work done is

$$\begin{aligned}
W &= \lim_{\Delta y \to 0} \sum 88200\pi(10-y)\ \Delta y \\
&= 88200\pi \int_0^{10} 10 - y\ dy \\
&= 88200\pi\left(10y - \frac{y^2}{2}\bigg|_0^{10}\right) \\
&\doteq 4.41\pi \times 10^6 J \\
&\doteq 1.39 \times 10^7 J.
\end{aligned}$$

VII.

(A) Show that for any constant c, $y = x^2 + \frac{c}{x^2}$ is a solution of the differential equation

$$y' = 4x - \frac{2}{x}y.$$

Solution: For the given y, we have $y' = 2x - \frac{2c}{x^3}$. On the other hand,

$$4x - \frac{2}{x}y = 4x - \frac{2}{x}(x^2 + \frac{c}{x^2}) = 2x - \frac{2c}{x^3}$$

also. Hence y is a solution of the given differential equation.

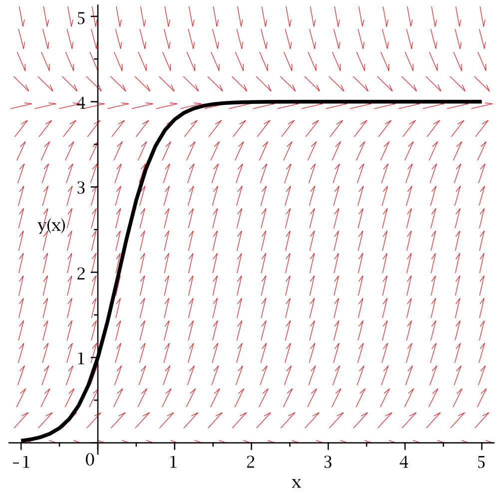

Figure 1: Slope field for $y' = y(4-y)$.

(B) All parts of this question refer to the differential equation

$$y' = y(4-y)$$

(1) Sketch the slope field of this equation, showing the slopes at points on the lines $y = 0, 1, 2, 3, 4, 5$.

Solution: (Note: The plot shows slope field arrows at more and different y-values than those given in the directions!)

(2) On your slope field, sketch the graph of the solution of the equation with $y(0) = 1$.

Solution: See the black curve in Figure 1.

(3) This is a separable equation; find the general solution and determine the constant of integration from the initial condition $y(0) = 1$.

Solution: To solve this equation (a *logistic equation*), we separate variables, use partial fractions to integrate the y-integral, then solve for y:

$$\begin{aligned}
\int \frac{dy}{y(4-y)} &= \int \, dx \\
\int \frac{1/4}{y} + \frac{1/4}{4-y} \, dy &= \int \, dx \\
\ln\left|\frac{y}{4-y}\right| &= 4x + c \\
\frac{y}{4-y} &= ke^{4x} \qquad \text{where } k = \pm e^c \\
y &= \frac{4ke^{4x}}{1+ke^{4x}} \cdot \frac{\frac{1}{k}e^{-4x}}{\frac{1}{k}e^{-4x}} \\
&= \frac{4}{1+me^{-4x}} \qquad \text{where } m = 1/k.
\end{aligned}$$

To get $y(0) = 1$, we want $1 = \frac{4}{1+m}$, so $m = 3$.

(C) Find the general solutions of the following differential equations

(1) $y' = \dfrac{y}{x(x+1)}$

Solution: Using partial fractions on the x-integral, we have

$$\begin{aligned} \int \frac{dy}{y} &= \int \frac{dx}{x(x+1)} \\ \ln|y| &= \ln|x| - \ln|x+1| + c \\ &= \ln\left|\frac{x}{x+1}\right| + c \\ y &= k\left|\frac{x}{x+1}\right| \qquad \text{where } k = \pm e^c. \end{aligned}$$

(2) $y' = \dfrac{\sqrt{1-x^2}}{e^{2y}}$.

Solution:

$$\begin{aligned} \int e^{2y}\, dy &= \int \sqrt{1-x^2}\, dx \\ \frac{1}{2}e^{2y} &= \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x) + c \qquad \text{by \# 30 in the table} \\ e^{2y} &= x\sqrt{1-x^2} + \sin^{-1}(x) + c \\ y &= \frac{1}{2}\ln(x\sqrt{1-x^2} + \sin^{-1}(x) + c). \end{aligned}$$

(D) Newton's Law of Cooling states that the rate at which the temperature of an object changes is proportional to the difference between the object's temperature and the surrounding temperature. A hot cup of tea with temperature 100°C is placed on a counter in a room maintained at constant temperature 20°C. Ten minutes later the tea has cooled to 76°C. How long will it take to cool off to 45°C? (Express Newton's Law as a differential equation, solve it for the temperature function, then use that to answer the question.)

Solution: Newton's Law as a differential equation gives $T' = k(T-20)$ here. Hence

$$\begin{aligned} \int \frac{dT}{T-20} &= \int k\, dt \\ \ln|T-20| &= kt + c \\ T &= 20 + me^{kt} \qquad \text{where } m = \pm e^c. \end{aligned}$$

Let $t = 0$ be the time when the cup of tea is placed on the counter. We know $T(0) = 100$ so $m = 80$. Then $T(10) = 76$, so $76 = 20 + 80e^{10k}$, and $k = \frac{\ln(56/80)}{10} \doteq -.0357$. Finally, we want to solve for the t when $45 = 20 + 80e^{-.0357t}$. This is $t = \frac{\ln(25/80)}{-.0357} \doteq 32.6$ minutes.