Mathematics 136, section 2 – Advanced Placement Calculus
Solutions for Practice Problems for Exam 2
October 31, 2009

Sample Exam – Note the real exam will contain different questions, and possibly different types of questions. The actual exam will be about this length.

I. In an alternate universe, the "super-attractive" force exerted by two objects of mass m_1, m_2 is *directly* proportional to the square of the distance between them: $F = Sm_1m_2r^2$, where S is a (*positive*) constant and r is the distance between the masses. Two 1kg masses are fixed at points $x = -1$ and $x = 3$ respectively along a straight line in this alternate universe. At what location x along that line should a third 2kg mass be placed to *minimize* the sum of the super-attractive forces exerted on it by the two unit masses?

Solution: The total force felt by a 2kg mass placed at x is

$$F(x) = 2S(x+1)^2 + 2S(x-3)^2.$$

We have $F'(x) = 4S(x+1+x-3) = 4S(2x-2)$. This is zero when $x = 1$. Since the second derivative is $F''(x) = 8S > 0$, this is a minimum by the second derivative test. *Comment: This is probably one of the easiest problems of this type that we have seen, once the function to be minimized is set up. Also be prepared for more difficult problems like some of the suggested practice problems on the review sheet.*

II. (15) Evaluate $\lim_{x\to\infty}\left(1-\frac{3}{x}\right)^x$.

Solution: This is a 1^∞ indeterminate form. So as in several problems from Problem Set 4, we take logarithms first, then rearrange to 0/0 form:

$$\begin{aligned}\ln\left(\left(1-\frac{3}{x}\right)^x\right) &= x\ln\left(1-\frac{3}{x}\right)\\ &= \frac{\ln\left(1-\frac{3}{x}\right)}{\frac{1}{x}}\end{aligned}$$

Now apply L'Hopital's Rule. The ratio of the derivatives is

$$\frac{\frac{1}{1-\frac{3}{x}}\cdot\frac{3}{x^2}}{\frac{-1}{x^2}} = \frac{-3}{1-\frac{3}{x}}.$$

In the limit as $x\to\infty$, this goes to -3. Therefore, $\lim_{x\to\infty}\left(1-\frac{3}{x}\right)^x = e^{-3}$.

III. (10) In a car moving at 90 ft/sec, the driver suddenly saw an obstacle 400 feet ahead and braked to a stop in 10 seconds. The car's velocity was recorded by a sensor in its onboard computer every two seconds:

t (*sec*)	0	2	4	6	8	10
$v(t)$ (ft/sec)	90	75	50	25	5	0

Can you say for sure from the information here whether the car hit the obstacle or not? Explain, using the left- and right-hand Riemann sums for $v(t)$.

Solution: The total distance traveled is $s = \int_0^{10} v(t)\, dt$ (feet). The left-hand Riemann sum approximation with $n = 5$ has $\Delta t = 2$, so

$$s \doteq v(0)\Delta t + \cdots + v(8)\Delta t = 180 + 150 + 100 + 50 + 10 = 490.$$

The right-hand Riemann sum approximation is

$$s \doteq v(2)\Delta t + \cdots + v(10)\Delta t = 150 + 100 + 50 + 10 + 0 = 310.$$

Since $v(t)$ is (apparently) decreasing on the whole interval, the left-hand sum is an overestimate and the right-hand sum is an underestimate. Since 400 is included in the interval between the two estimates, we *cannot say for sure*. It is possible, but the car may also have stopped short of the obstacle. It would depend on what happened between the times at which the velocity values are known.

IV.

A) (10) Using the definition of the definite integral (not the Evaluation Theorem), determine $\int_1^2 x^2 + 3x + 1\, dx$. (Possibly useful information:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.)$$

Check your answer using the Evaluation Theorem.

Solution: We subdivide $[1,2]$ into n equal subintervals with $\Delta x = 1/n$. The right endpoints are $x_i = 1 + i/n$ so setting up the Riemann sum and using the summation formulas above,

$$\begin{aligned}
\int_1^2 x^2 + 3x + 1\, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left((1+i/n)^2 + 3(1+i/n) + 1\right)(1/n) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(5 + 5i/n + i^2/n^2\right)(1/n) \\
&= \lim_{n\to\infty} \left((5/n)\sum_{i=1}^{n} 1 + (5/n^2)\sum_{i=1}^{n} i + (1/n^3)\sum_{i=1}^{n} i^2\right) \\
&= \lim_{n\to\infty} \left(5 + \frac{5}{2} + \frac{5}{2n} + \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) \\
&= \frac{47}{6}.
\end{aligned}$$

Checking by the Evaluation Theorem:

$$\begin{aligned}
\int x^2 + 3x + 1 \; dx &= \frac{x^3}{3} + \frac{3x^2}{2} + x \Big|_1^2 \\
&= \frac{8}{3} + 6 + 2 - \frac{1}{3} - \frac{3}{2} - 1 \\
&= \frac{16 + 36 + 12 - 2 - 9 - 6}{6} \\
&= \frac{47}{6}.
\end{aligned}$$

B) (10) What is $\frac{d}{dx}\int_1^{x^2} \frac{\sin(t)}{t} \; dt$?

Solution: By the first part of the FTC and the chain rule, this is

$$\frac{\sin(x^2)}{x^2} \cdot 2x = \frac{2\sin(x^2)}{x}.$$

V. Methods of integration. In B,C,D you may use the table of integrals provided.

A) (10) Using integration by parts, show that

$$\int x^n \cos(ax) \; dx = \frac{x^n}{a}\sin(ax) - \frac{n}{a}\int x^{n-1}\sin(ax) \; dx.$$

Solution: Let $u = x^n$ and $dv = \cos(ax) \; dx$. Then $du = nx^{n-1} \; dx$ and $v = \frac{1}{a}\sin(ax)$. So applying the parts formula $\int u \; dv = uv - \int v \; du$:

$$\int x^n \cos(ax) \; dx = \frac{x^n}{a}\sin(ax) - \frac{n}{a}\int x^{n-1}\sin(ax) \; dx$$

as desired.

B) (10) $\int \tan(\sqrt{x})/\sqrt{x} \; dx$

Solution: Make the preliminary substitution $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}} \; dx$, so by # 12 in the table, or another substition, the integral is

$$2\int \tan(u) \; du = 2\ln|\sec(u)| + C = 2\ln|\sec(\sqrt{x})| + C.$$

C) (10) $\int (3x+2)/(x^2+8x+7) \; dx$

Solution: We apply the partial fractions method here since $x^2+8x+7 = (x+1)(x+7)$:

$$\begin{aligned}
\frac{3x+2}{(x+1)(x+7)} &= \frac{A}{x+1} + \frac{B}{x+7} \\
3x + 2 &= A(x+7) + B(x+1)
\end{aligned}$$

Setting $x = -1$ gives $-1 = 6A$ so $A = \frac{-1}{6}$. Then $x = -7$ gives $-19 = -6B$, so $B = \frac{19}{6}$. The integral is

$$\int \frac{\frac{-1}{6}}{x+1} + \frac{\frac{19}{6}}{x+7}\, dx = \frac{-1}{6} \ln|x+1| + \frac{19}{6} \ln|x+7| + C.$$

D) (10) $\int (4 - x^2)^{-3/2}\, dx$

Solution: This is the form of # 38 in the table with $a = 4$. So the integral is

$$\frac{x}{4\sqrt{4 - x^2}} + C.$$