Mathematics 136 – Advanced Placement Calculus
Discussion 3 – Related Rates "Story Problems" with solutions
September 23, 2009

Background and Example Problem

For the next few days, we will look at several applications of derivatives. We will first consider the *related rates problems*. Here is a typical example:

A baseball diamond is a square which has side 90 feet. A batter hits the ball and runs toward first base at 24 ft/sec. How fast is his distance from second base changing when he is halfway to 1st base?

To solve problems like this, a good general strategy is to

1. identify the important quantities in the problem.
2. identify the unknown quantity that you are asked to solve for.
3. draw a picture representing the situation at one or more times, and think about what changes with time. (Note: detailed drawings of the real-world situation are unnecessary; schematic versions are fine. See below.)
4. find an equation relating the quantities, valid at all times.
5. differentiate the equation with respect to t (this often requires a process like *implicit differentiation.*
6. substitute given values and solve for the unknown.

The important quantities are the distance x the runner has traveled from home plate, the distance $90 - x$ left to first base, and 90 feet from second base to first base, and the distance c from second base to the runner. Both x and c change with time, and we see $\frac{dx}{dt} = 24$ feet/sec. For our example problem, the picture is shown in Figure 1. We want to know the rate at which c is changing at a particular time, so a value of $\frac{dc}{dt}$. From the figure we see that the runner, first base, and second base are the corners of a right triangle. So the Pythagorean Theorem gives us a relation that is valid at all times:

$$c^2 = (90 - x)^2 + 90^2.$$

In this equation, c and x are really functions of the time, t. Hence we can differentiate through on both sides with respect to t to obtain:

$$2c\frac{dc}{dt} = 2(90 - x) \cdot \left(-\frac{dx}{dt}\right).$$

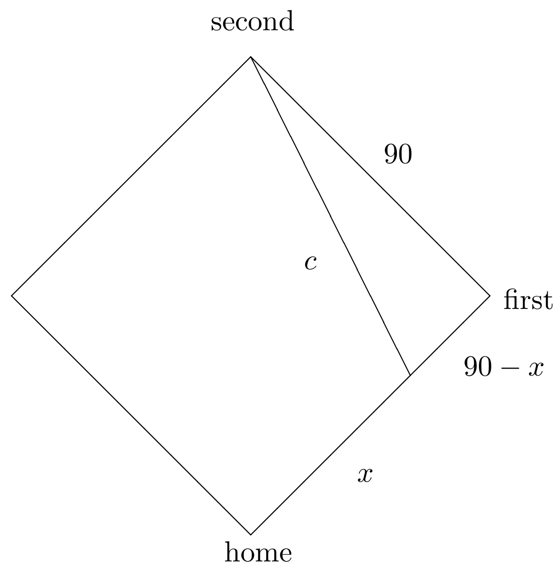

Figure 1: The runner is x feet from home and c feet from second

When the runner is halfway to first base, $x = 45$, so $c^2 = 45^2 + 90^2 = 45^2 \cdot 5$, and $c = 45\sqrt{5}$. Hence

$$\begin{aligned} 2 \cdot 45\sqrt{5} &= -2(90-45)\cdot(24) \\ \text{so } \frac{dx}{dt} &= -\frac{25}{\sqrt{5}} \\ &\doteq -10.7\text{ft/sec.} \end{aligned}$$

Note that this says the distance to second is *decreasing* at the rate of 10.7 feet per second.

Discussion Problems

A. In the situation of the Sample Problem, how fast is the distance to the runner from *third* base changing when the runner is halfway to first? Is that distance increasing or decreasing?

   Solution: Refer to the diagram above. Third base, home plate, and the runner's current position form a right triangle with sides $90, x, c$. So by the Pythagorean Theorem, $c^2 = 90^2 + x^2$ at all times. Then differentiation with respect to t gives:

$$2c\frac{dc}{dt} = 2x\frac{dx}{dt}.$$

   When the runner is half way to first base,

$$2(45\sqrt{5})\frac{dc}{dt} = 2(45)\frac{dx}{dt},$$

so since $\frac{dx}{dt} = 24$,

$$\frac{dc}{dt} = \frac{24}{\sqrt{5}} \doteq 10.7 \text{ ft/sec}.$$

The distance from the runner to third base is increasing at a rate of roughly 10.7 feet per second.

B. An oil slick has the shape of a circular cylinder of constant volume 100 cubic meters. The radius of the slick is increasing at the rate of 10 meters per hour. How fast is the height of the slick changing when the radius is 100 meters?

Solution: The volume of a cylinder of height h and base radius r is $V = \pi r^2 h$. This is the relation that is true for all times. Taking derivatives with respect to t gives:

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi r h \frac{dr}{dt}. \tag{1}$$

The volume of the oil in the slick is constant, so $\frac{dV}{dt} = 0$. We want $\frac{dh}{dt}$ when $r = 100$. At that time, $100 = \pi(100)^2 h$, so $h = \frac{1}{100\pi}$. Then substituting all this into 1, we get

$$0 = \pi(100)^2 \frac{dh}{dt} + 2\pi \cdot 100 \cdot \frac{1}{100\pi} \cdot 10.$$

Solving for $\frac{dh}{dt}$,

$$\frac{dh}{dt} = -\frac{20}{(100)^2\pi} \doteq -6.3 \times 10^{-4}$$

(meters per hour). Note that the height of the slick must decrease as it spreads out in order to keep the volume constant.

C. Gravel is being dumped from a conveyor belt at the rate of 30 cubic feet per minute. The pile of gravel formed is a right circular cone whose height and radius are equal at all times. How fast is the height of the pile increasing when the pile is 10 feet high?

Solution: The volume of a cone of base radius r and height h is $V = \frac{\pi r^2 h}{3}$. Since we are given that $h = r$ at all times, we can rewrite this as $V = \frac{\pi h^3}{3}$. (Note that we could also have written this in terms of r alone. Things come out the same either way.) The derivative with respect to time is

$$\frac{dV}{dt} = \pi h^2 \frac{dh}{dt}.$$

The volume is increasing at 30 cubic feet per minute, so $\frac{dV}{dt} = 30$, and we want $\frac{dh}{dt}$ when $h = 10$. So

$$30 = 100\pi \frac{dh}{dt},$$

and $\frac{dh}{dt} = \frac{3}{10\pi} \doteq .095$ feet per minute.

D. A woman 5 feet tall walks at 4 feet per second straight toward a street lamp 20 feet above ground level. The lamp casts a shadow behind the woman as she walks. How fast is the tip of the woman's shadow moving?

Solution: We assume the woman and the lamp post are vertical. Then the *big triangle* formed by the lamp, the base of the lamp post, and the tip of the shadow, and the *small triangle* formed by the top of the woman's head, the bottom of her feet, and the tip of the shadow are *similar* at all times. Call the distance from the woman to the lamp x, and the length of the shadow s. Then we have by proportionality

$$\frac{20}{s+x} = \frac{5}{s}.$$

Cross-multiplying and simplifying,

$$20s = 5s + 5x, \text{ which implies } 3s = x.$$

Then differentiating with respect to t, $3\frac{ds}{dt} = \frac{dx}{dt}$. We are given that $\frac{dx}{dt} = 4$, so $\frac{ds}{dt} = \frac{4}{3}$ feet per second.