MATH 136 – Advanced Placement Calculus
Solutions for Resubmitted Problems – Problem Set 4
October 26, 2009

4.3/12. (a) We have $f(x) = \cos^2(x) - 2\sin(x)$, so $f'(x) = -2\cos(x)\sin(x) - 2\cos(x) = -2\cos(x)(\sin(x) + 1)$. So $f'(x) = 0$ at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ in the interval $[0, 2\pi]$. Since $\sin(x) + 1 \geq 0$, the sign of $-\cos(x)$ gives the sign of f' and this is negative on $\left[0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right]$, positive on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

(b) By the first derivative test, $f(x)$ has a local maximum at $x = \frac{3\pi}{2}$ and a local minimum at $x = \frac{\pi}{2}$.

(c) The second derivative is

$$\begin{aligned} f''(x) &= 2\sin^2(x) - 2\cos^2(x) + 2\sin(x) \\ &= 2\sin^2(x) - 2(1 - \sin^2(x)) + 2\sin(x) \\ &= 4\sin^2(x) + 2\sin(x) - 2 \\ &= 2(\sin(x) + 1)(2\sin(x) - 1) \end{aligned}$$

Now, $\sin(x) = -1$ at $x = \frac{3\pi}{2}$, while $\sin(x) = \frac{1}{2}$ at $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$. Of these, only $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$ are inflection points because $f''(x)$ does not change sign at $x = \frac{3\pi}{2}$. (Note that $f'(x)$ has a double root at $x = \frac{3\pi}{2}$. This means that the graph of $f(x)$ is very "flat" around the maximum, but the concavity does not change there.)

4.4/14. The limit $\lim_{x\to\infty} \frac{(\ln(x))^2}{x}$ is indeterminate of the form ∞/∞. Applying L'Hopital a first time, we examine

$$\lim_{x\to\infty} \frac{2\ln(x)\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2\ln(x)}{x}.$$

Since this is still indeterminate of the form ∞/∞, we apply L'Hopital's Rule a second time:

$$\lim_{x\to\infty} \frac{\frac{2}{x}}{1} = 0.$$

Hence the original limit is 0 also.

4.6/12. Let x be the side of the base of the box and y be the height. We have $32000 = x^2 y$, so $y = \frac{32000}{x^2}$. The surface area of the bottom and the sides (no top) is

$$S = x^2 + 4xy = x^2 + 4x\frac{32000}{x^2} = x^2 + \frac{168000}{x}.$$

Now we differentiate with respect to x and set the derivative equal to zero to find the critical numbers:

$$0 = S' = 2x - \frac{168000}{x^2} = 0 \Rightarrow x^3 = 64000$$

so $x = 40$. Substituting back, $y = \frac{32000}{1600} = 20$. This is a *minimum* of the S function (and the material needed for the box), since $S'' = 2 + \frac{336000}{x^3} > 0$ for all $x > 0$. The second derivative test shows that S has a local minimum at $x = 40$, and this is a global minimum since it is the only critical number for $x > 0$ and S is differentiable for all $x > 0$.

4.8/30. Given $f''(x) = 8x^3 + 5$, and $f'(1) = 8$, $f(1) = 0$. Integrating once, $f'(x) = 2x^4 + 5x + c$. Then substituting $x = 1$ gives $8 = f'(1) = 2 + 5 + c$. Hence $c = 1$. Integrating again, $f(x) = \frac{2}{5}x^5 + \frac{5}{2}x^2 + x + d$. Substituting $x = 1$ again, $\frac{2}{5} + \frac{5}{2} + 1 + d = 0$. Hence $d = \frac{-(4+25+10)}{10} = \frac{-39}{10}$. Then

$$f(x) = \frac{2}{5}x^5 + \frac{5}{2}x^2 + x - \frac{39}{10}.$$