Mathematics 136, section 2 – AP Calculus
Solutions for Final Exam Practice Questions
December 14, 2009

I.

A) The graph $y = 2f(x-2)+1$ would be obtained by shifting the graph $y = f(x)$ to the right by 2 units, stretching vertically by a factor of 2, then shifting the resulting graph up by 1 unit.

B) The graph $y = -f(-x)$ would be obtained by reflecting the graph $y = f(x)$ across the y-axis, then reflecting the resulting graph across the x-axis.

II.

A) For problems of this kind, look for where the given graph has horizontal tangents (zeroes of f'), and positive and negative slopes. I will look for qualitative information rather than any quantitative estimates of actual slope values.

B) These are the reverse of question A. When $f(x)$ is > 0, the antiderivative $g(x)$ is increasing. When $f(x) < 0$, the antiderivative $g(x)$ is decreasing. $g(x)$ will have local maxima at x-values where f changes from positive to negative, and local minima where f changes from negative to positive (First Derivative Test).

III.

A) The derivative $f'(a)$ is:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

provided that the limit exists.

B) The definite integral of $f(x)$ over $[a, b]$ is the limit of the Riemann sums

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided that the limit exists (independently of the choice of the x_i^*).

C) Let $f(x)$ be continuous on $[a, b]$. Then

1) the function

$$G(x) = \int_a^x f(t)\ dt$$

is an antiderivative of $f(x)$ on (a, b) (that is $G'(x) = f(x)$ for all x in (a, b)).

2) If $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

IV.

A) The plate is a 6×4 rectangle. Subdivide it into 6 vertical strips of width $\Delta x = 1$ at the given x-values. Since the density of the metal does not depend on y, we can estimate the density in any vertical strip by a density value from the table on that strip times the area of the strip. Using the left endpoints, for instance, this gives

$$M \doteq 200 \cdot 1 \cdot 4 + 190 \cdot 1 \cdot 4 + 170 \cdot 1 \cdot 4 + 140 \cdot 1 \cdot 4 + 100 \cdot 1 \cdot 4 + 90 \cdot 1 \cdot 4 = 3560$$

grams.

B) Since we used the left endpoints, and the density appears to be decreasing left-to-right across the plate, this estimate is probably *too large*.

V.

A) Use a u-substitution on each term:

$$\begin{aligned}\int_0^1 \cos(\pi x) - x^2 e^{x^3}\ dx &= \frac{1}{\pi}\sin(\pi x) - \frac{1}{3}e^{x^3}\Big|_0^1 \\ &= 0 - \frac{1}{3}e + \frac{1}{3} \\ &= \frac{1}{3}(1-e).\end{aligned}$$

B) We integrate by parts *twice* letting u be the power of x each time:

$$\begin{aligned}\int x^2 \sin(3x)\ dx &= \frac{-x^2}{\cos(3x)} + \frac{2}{3}\int x\cos(3x)\ dx \\ &= \frac{-x^2}{3}\cos(3x) + \frac{2}{3}\left(\frac{x}{3}\sin(3x) - \frac{1}{3}\int \sin(3x)\ dx\right) \\ &= \frac{-x^2}{3}\cos(3x) + \frac{2x}{9}\sin(3x) + \frac{2}{27}\cos(3x) + C\end{aligned}$$

C) The partial fractions look like

$$\frac{x}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}.$$

Clearing denominators gives:

$$x = A(x^2+1) + (Bx+C)(x+3),$$

or

$$x = (A+B)x^2 + (3B+C)x + (3C+A).$$

Hence $A+B=0$, $3B+C=1$, and $3C+A=0$. From the last equation, $A=-3C$. So the first equation becomes $B-3C=0$ and the second is $3B+C=1$. Then

$(B-3C)+3(3B+C)=10B=3$, so $B=\frac{3}{10}$, $C=\frac{1}{10}$, and $A=\frac{-3}{10}$. Integrating the partial fractions gives:

$$\begin{aligned}\int \frac{\frac{-3}{10}}{x+3}+\frac{\frac{3}{10}x+\frac{1}{10}}{x^2+1}\ dx &= \frac{-3}{10}\int \frac{1}{x+3}\ dx+\frac{3}{10}\int \frac{x}{x^2+1}\ dx+\frac{1}{10}\int \frac{1}{x^2+1}\ dx\\ &= \frac{-3}{10}\ln|x+3|+\frac{3}{20}\ln(x^2+1)+\frac{1}{10}\tan^{-1}(x)+C.\end{aligned}$$

D) Completing the square gives

$$\int \frac{dx}{\sqrt{(x-8)^2+1}}$$

We let $x-8=\tan\theta$, so $dx=\sec^2\theta\ d\theta$, and the integral becomes

$$\int \frac{\sec^2\theta\ d\theta}{\sqrt{\tan^2\theta+1}}=\int \sec\theta\ d\theta$$

Using the appropriate entry from the integral table, this gives

$$=\ln|\sec\theta+\tan\theta|+C=\ln|\sqrt{(x-8)^2+1}+x-8|+C.$$

VI.

A) The cross-sections by planes $x=$ constant are washers, so the volume is computed by the integral

$$V=\int_0^2 \pi(4-x^2)^2\ dx=\frac{256\pi}{15}.$$

B) The cross-sections by planes $y=$ constant are washers with outer radius 3 and inner radius $3-\sqrt{4-y}$. So the volume is

$$V=\int_0^4 \pi(3)^2-\pi(3-\sqrt{4-y})^2\ dy=24.$$

C) The coordinates of the centroid are

$$\overline{x}=\frac{\int_0^2 x(4-x^2)\ dx}{\int_0^2 4-x^2\ dx}=\frac{3}{4},$$

and

$$\overline{y}=\frac{\int_0^2 \frac{1}{2}(4-x^2)^2\ dx}{\int_0^2 4-x^2\ dx}=\frac{8}{5}.$$

VII.

A) Graph omitted – the curves $xy = c$ are rectangular hyperbolas with asymptotes along the coordinate axes.
B) Separating variables and integrating:

$$\int \frac{dy}{y} = \int x\ dx$$
$$\ln |y| = \frac{x^2}{2} + C$$
$$y = ke^{x^2/2}$$

where $k = \pm e^C$. To get $y(0) = 4$, we want $k = 4$, so the solution is

$$y = 4e^{x^2/2}.$$

VIII.
A) The differential equation is $\frac{dT}{dt} = k(T - A)$.
B) The solution, after separating and integrating is

$$T = A + ke^{kt},$$

C) Using the result from B with $A = 70$, we have $T(0) = 200$, so $k = 130$. Then at $t = 5$, $170 = 70 + 130e^{5k}$, so

$$k = \frac{1}{5} \ln \left(\frac{10}{13} \right) = -.0525.$$

Then we want the time t when $120 = 70 + 130e^{(-.0525)t}$, so

$$t = \frac{\ln \left(\frac{50}{130} \right)}{-.0525} \doteq 18.2$$

minutes.

IX. Put the top of the punch bowl at $y = 0$. The work done is

$$W = \int_{-0.5}^{0} \pi(\sqrt{(.5)^2 - y^2})^2 \cdot 950 \cdot 9.8(0 - y)\ dy \doteq 457$$

Joules. (Note: 1J is 1 $kg \cdot m^2/sec^2$.)

X.
A) Since $0 < \frac{e}{\pi} < 1$, the geometric series converges to

$$\frac{1}{1 - \frac{e}{\pi}} = \frac{\pi}{\pi - e}.$$

B) For each $k \geq 0$, the kth derivative of $f(x) = e^{2x}$ at $a = 0$ is $f^{(k)}(0) = 2^k$. So the Taylor series is

$$\sum_{k=0}^{\infty} \frac{2^k x^k}{k!} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{6!} + \cdots.$$

C) The fifth degree Taylor polynomial of $\sin(x)$ at $a = 0$ is

$$p_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

At $x = .6$, this gives

$$\sin(.6) \doteq p_5(.6) = .564648.$$

The actual value of $\sin(.6)$ is about .564642, so the absolute error is about .000006.