Mathematics 136 – AP Calculus
Makeup Exam 2 Solutions – November 14, 2003

I. (20) In an alternate universe, the "super-attractive" force exerted by two objects of mass m_1, m_2 is *directly* proportional to the square of the distance between them: $F = Sm_1m_2r^2$, where S is a positive constant and r is the distance between the masses. Four 1kg masses are fixed at points $x = -12, -4, 2, 7$ respectively along a straight line in this alternate universe. At what location x along that line should another 2kg mass be placed to *minimize* the sum of the super-attractive forces exerted on it by the four unit masses?

Solution: The distance between a and b along the number line is $|b-a| = |a-b|$. When we square, the absolute value can be disregarded, since the sign does not matter. Hence the total force exerted by the four masses on the 2kg mass at x is

$$\begin{aligned} F(x) &= 2S((x-(-12))^2 + (x-(-4))^2 + (x-2)^2 + (x-7)^2) \\ &= 2S((x+12)^2 + (x+4)^2 + (x-2)^2 + (x-7)^2) \end{aligned}$$

To minimize F we differentiate and set to zero:

$$0 = F'(x) = 2S(2(x+12) + 2(x+4) + 2(x-2) + 2(x-7)) = 4S(4x+7)$$

This is zero if and only if $x = -7/4$. (This is clearly a local minimum for F, since $F'' = 16S > 0$.)

II. In a car moving at 90 ft/sec, the driver suddenly saw an obstacle 400 feet ahead and braked to a stop in 10 seconds. The car's velocity was recorded by a sensor in its onboard computer every two seconds:

t (sec)	0	2	4	6	8	10
$v(t)$ (ft/sec)	90	75	50	25	5	0

A) (10) Can you say for sure from the information here whether the car hit the obstacle or not? Explain, using the left- and right-hand Riemann sums for $v(t)$.

Solution: The left-hand Riemann sum gives

$$LHS = 2 \cdot 90 + 2 \cdot 75 + 2 \cdot 50 + 2 \cdot 25 + 2 \cdot 5 = 490$$

The right-hand Riemann sum gives

$$RHS = 2 \cdot 75 + 2 \cdot 50 + 2 \cdot 25 + 2 \cdot 5 + 2 \cdot 0 = 310$$

Since the velocity is apparently decreasing on the whole interval, we expect the LHS is an *overestimate* of the actual distance travelled and the RHS is an *underestimate* of the actual distance. This says $310 \leq$ actual distance ≤ 490. But it is not possible to say for sure from this information whether a collision occurred.

B) (10) From the given information, what is your best estimate about whether the car hit the obstacle?

Solution: The average $(RHS + LHS)/2 = 400$ exactly. There is a good chance the car did hit or just brush the obstacle.

III.

A) (10) State the first and second parts of the Fundamental Theorem of Calculus.

Solution: See text or class notes.

B) (10) The following graph shows $y = f(x)$ for some function. Sketch the graph of the antiderivative $F(x)$ of $g(x)$ with $F(x) = 0$. What is $F(4)$?

Solution: The graph of F should have a linear segment from $x = 0$ to $x = 1$ of slope 1, so $F(1) = 1$. Then between $x = 1$ and $x = 2$, the graph is part of a parabola opening down. The areas above and below the x-axis between $x = 1$ and $x = 2$ exactly balance out so $F(2) = F(1) = 1$. Between $x = 2$ and $x = 3$, the graph is part of a parabola opening up; the area below the axis is $-1/2$ (a triangle of base 1 and height 1). So $F(3) - F(2) = -1/2$ and $F(3) = 1/2$. Then the rest of the graph to $x = 4$ is part of a parabola opening down and $F(4) - F(3) = -1/2$, so $F(4) = 0$.

IV. Methods of integration. In B,C,D you may use the table of integrals.

A) (10) Using integration by parts, show that

$$\int x^n \cos(ax)\, dx = \frac{x^n}{a} \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax)\, dx.$$

Solution: Let $u = x^n$, $dv = \cos(ax)\, dx$. Then $du = nx^{n-1}$ and $v = \frac{1}{a}\sin(ax)$, so using the integration by parts formula,

$$\int x^n \cos(ax)\, dx = \int u\, dv = uv - \int v\, du = \frac{x^n}{a} \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax)\, dx$$

which is what we had to show.

B) (10) $\int \tan(\sqrt{x})/\sqrt{x}\, dx$

Solution: (This is a u-substitution form.) Let $u = \sqrt{x} = x^{1/2}$. Then $du = \frac{1}{2\sqrt{x}}$ so the $\sqrt{x}$ in the denominator of the integrand is part of du. We have, using # 7 in the table (or by another substitution)

$$\begin{aligned}\int \tan(\sqrt{x})/\sqrt{x}\, dx &= 2 \int \tan(u)\, du \\ &= -2\ln(\cos(u)) + C \\ &= -2\ln(\cos(\sqrt{x})) + C\end{aligned}$$

C) (10) $\int (3x+2)/(x^2+8x+7)\ dx$

Solution: The quadratic in the denominator factors as $x^2+8x+7=(x+1)(x+7)$ (or, completing square, get $(x+4)^2-9$ and *the negative sign* says there are real roots). Hence we want to use #27 in the table with $a=-1, b=-7, c=3, d=2$. The result is

$$\frac{1}{6}\left(((-1)(3)+2)\ln|x+1|-((-7)(3)+2)\ln|x+7|\right)+C$$

which simplifies to

$$\frac{1}{6}\left(-\ln|x+1|+19\ln|x+7|\right)+C$$

D) (10) $\int (4-x^2)^{-3/2}\ dx$

Solution: (This is a trigonometric substitution form.) Let $x=2\sin(\theta)$, so $dx=2\cos(\theta)d\theta$. Then the integral becomes

$$\begin{aligned}\int (4-x^2)^{-3/2}\ dx &= 2\int (4\cos^2(\theta))^{-3/2}\cos(\theta)\ d\theta\\ &= \frac{1}{4}\int \cos^{-3}(\theta)\cos(\theta)\ d\theta\\ &= \frac{1}{4}\int \cos^{-2}(\theta)\ d\theta\\ &= \frac{1}{4}\int \sec^2(\theta)\ d\theta\\ &= \frac{1}{4}\tan(\theta)+C\end{aligned}$$

To convert back to x, we use $\sin(\theta)=x/2$, so $\cos(\theta)=\sqrt{4-x^2}/2$, and $\tan(\theta)=\sin(\theta)/\cos(\theta)=x/\sqrt{4-x^2}$. The final answer is

$$\int (4-x^2)^{-3/2}\ dx=\frac{1}{4}\frac{x}{\sqrt{4-x^2}}+C$$