

Mathematics 136 – AP Calculus  
Makeup Exam 2 Solutions – November 14, 2003

I. (20) In an alternate universe, the “super-attractive” force exerted by two objects of mass  $m_1, m_2$  is *directly* proportional to the square of the distance between them:  $F = Sm_1m_2r^2$ , where  $S$  is a positive constant and  $r$  is the distance between the masses. Four 1kg masses are fixed at points  $x = -12, -4, 2, 7$  respectively along a straight line in this alternate universe. At what location  $x$  along that line should another 2kg mass be placed to *minimize* the sum of the super-attractive forces exerted on it by the four unit masses?

*Solution:* The distance between  $a$  and  $b$  along the number line is  $|b - a| = |a - b|$ . When we square, the absolute value can be disregarded, since the sign does not matter. Hence the total force exerted by the four masses on the 2kg mass at  $x$  is

$$\begin{aligned} F(x) &= 2S((x - (-12))^2 + (x - (-4))^2 + (x - 2)^2 + (x - 7)^2) \\ &= 2S((x + 12)^2 + (x + 4)^2 + (x - 2)^2 + (x - 7)^2) \end{aligned}$$

To minimize  $F$  we differentiate and set to zero:

$$0 = F'(x) = 2S(2(x + 12) + 2(x + 4) + 2(x - 2) + 2(x - 7)) = 4S(4x + 7)$$

This is zero if and only if  $x = -7/4$ . (This is clearly a local minimum for  $F$ , since  $F'' = 16S > 0$ .)

II. In a car moving at 90 ft/sec, the driver suddenly saw an obstacle 400 feet ahead and braked to a stop in 10 seconds. The car’s velocity was recorded by a sensor in its onboard computer every two seconds:

$t$ (sec)	0	2	4	6	8	10
$v(t)$ (ft/sec)	90	75	50	25	5	0

A) (10) Can you say for sure from the information here whether the car hit the obstacle or not? Explain, using the left- and right-hand Riemann sums for  $v(t)$ .

*Solution:* The left-hand Riemann sum gives

$$LHS = 2 \cdot 90 + 2 \cdot 75 + 2 \cdot 50 + 2 \cdot 25 + 2 \cdot 5 = 490$$

The right-hand Riemann sum gives

$$RHS = 2 \cdot 75 + 2 \cdot 50 + 2 \cdot 25 + 2 \cdot 5 + 2 \cdot 0 = 310$$

Since the velocity is apparently decreasing on the whole interval, we expect the LHS is an *overestimate* of the actual distance travelled and the RHS is an *underestimate* of the

actual distance. This says  $310 \leq \text{actual distance} \leq 490$ . But it is not possible to say for sure from this information whether a collision occurred.

- B) (10) From the given information, what is your best estimate about whether the car hit the obstacle?

*Solution:* The average  $(RHS + LHS)/2 = 400$  exactly. There is a good chance the car did hit or just brush the obstacle.

III.

- A) (10) State the first and second parts of the Fundamental Theorem of Calculus.

*Solution:* See text or class notes.

- B) (10) The following graph shows  $y = f(x)$  for some function. Sketch the graph of the antiderivative  $F(x)$  of  $f(x)$  with  $F(x) = 0$ . What is  $F(4)$ ?

*Solution:* The graph of  $F$  should have a linear segment from  $x = 0$  to  $x = 1$  of slope 1, so  $F(1) = 1$ . Then between  $x = 1$  and  $x = 2$ , the graph is part of a parabola opening down. The areas above and below the  $x$ -axis between  $x = 1$  and  $x = 2$  exactly balance out so  $F(2) = F(1) = 1$ . Between  $x = 2$  and  $x = 3$ , the graph is part of a parabola opening up; the area below the axis is  $-1/2$  (a triangle of base 1 and height 1). So  $F(3) - F(2) = -1/2$  and  $F(3) = 1/2$ . Then the rest of the graph to  $x = 4$  is part of a parabola opening down and  $F(4) - F(3) = -1/2$ , so  $F(4) = 0$ .

IV. Methods of integration. In B,C,D you may use the table of integrals.

- A) (10) Using integration by parts, show that

$$\int x^n \cos(ax) dx = \frac{x^n}{a} \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax) dx.$$

*Solution:* Let  $u = x^n$ ,  $dv = \cos(ax) dx$ . Then  $du = nx^{n-1}$  and  $v = \frac{1}{a} \sin(ax)$ , so using the integration by parts formula,

$$\int x^n \cos(ax) dx = \int u dv = uv - \int v du = \frac{x^n}{a} \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax) dx$$

which is what we had to show.

- B) (10)  $\int \tan(\sqrt{x})/\sqrt{x} dx$

*Solution:* (This is a  $u$ -substitution form.) Let  $u = \sqrt{x} = x^{1/2}$ . Then  $du = \frac{1}{2\sqrt{x}}$  so the  $\sqrt{x}$  in the denominator of the integrand is part of  $du$ . We have, using # 7 in the table (or by another substitution)

$$\begin{aligned} \int \tan(\sqrt{x})/\sqrt{x} dx &= 2 \int \tan(u) du \\ &= -2 \ln(\cos(u)) + C \\ &= -2 \ln(\cos(\sqrt{x})) + C \end{aligned}$$

C) (10)  $\int (3x + 2)/(x^2 + 8x + 7) dx$

*Solution:* The quadratic in the denominator factors as  $x^2 + 8x + 7 = (x + 1)(x + 7)$  (or, completing square, get  $(x + 4)^2 - 9$  and *the negative sign* says there are real roots). Hence we want to use #27 in the table with  $a = -1, b = -7, c = 3, d = 2$ . The result is

$$\frac{1}{6} (((-1)(3) + 2) \ln |x + 1| - ((-7)(3) + 2) \ln |x + 7|) + C$$

which simplifies to

$$\frac{1}{6} (-\ln |x + 1| + 19 \ln |x + 7|) + C$$

D) (10)  $\int (4 - x^2)^{-3/2} dx$

*Solution:* (This is a trigonometric substitution form.) Let  $x = 2 \sin(\theta)$ , so  $dx = 2 \cos(\theta) d\theta$ . Then the integral becomes

$$\begin{aligned} \int (4 - x^2)^{-3/2} dx &= 2 \int (4 \cos^2(\theta))^{-3/2} \cos(\theta) d\theta \\ &= \frac{1}{4} \int \cos^{-3}(\theta) \cos(\theta) d\theta \\ &= \frac{1}{4} \int \cos^{-2}(\theta) d\theta \\ &= \frac{1}{4} \int \sec^2(\theta) d\theta \\ &= \frac{1}{4} \tan(\theta) + C \end{aligned}$$

To convert back to  $x$ , we use  $\sin(\theta) = x/2$ , so  $\cos(\theta) = \sqrt{4 - x^2}/2$ , and  $\tan(\theta) = \sin(\theta)/\cos(\theta) = x/\sqrt{4 - x^2}$ . The final answer is

$$\int (4 - x^2)^{-3/2} dx = \frac{1}{4} \frac{x}{\sqrt{4 - x^2}} + C$$