I. (20) In an alternate universe, the “super-attractive” force exerted by two objects of mass \( m_1, m_2 \) is directly proportional to the square of the distance between them: \( F = Sm_1m_2r^2 \), where \( S \) is a positive constant and \( r \) is the distance between the masses. Four 1kg masses are fixed at points \( x = -12, -4, 2, 7 \) respectively along a straight line in this alternate universe. At what location \( x \) along that line should another 2kg mass be placed to minimize the sum of the super-attractive forces exerted on it by the four unit masses?

**Solution:** The distance between \( a \) and \( b \) along the number line is \( |b - a| = |a - b| \). When we square, the absolute value can be disregarded, since the sign does not matter. Hence the total force exerted by the four masses on the 2kg mass at \( x \) is

\[
F(x) = 2S((x - (-12))^2 + (x - (-4))^2 + (x - 2)^2 + (x - 7)^2)
\]

To minimize \( F \) we differentiate and set to zero:

\[
0 = F'(x) = 2S(2(x + 12) + 2(x + 4) + 2(x - 2) + 2(x - 7)) = 4S(4x + 7)
\]

This is zero if and only if \( x = -7/4 \). (This is clearly a local minimum for \( F \), since \( F'' = 16S > 0 \).)

II. In a car moving at 90 ft/sec, the driver suddenly saw an obstacle 400 feet ahead and braked to a stop in 10 seconds. The car’s velocity was recorded by a sensor in its onboard computer every two seconds:

<table>
<thead>
<tr>
<th>( t ) (sec)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v(t) ) (ft/sec)</td>
<td>90</td>
<td>75</td>
<td>50</td>
<td>25</td>
<td>5</td>
<td>0</td>
</tr>
</tbody>
</table>

A) (10) Can you say for sure from the information here whether the car hit the obstacle or not? Explain, using the left- and right-hand Riemann sums for \( v(t) \).

**Solution:** The left-hand Riemann sum gives

\[
LHS = 2 \cdot 90 + 2 \cdot 75 + 2 \cdot 50 + 2 \cdot 25 + 2 \cdot 5 = 490
\]

The right-hand Riemann sum gives

\[
RHS = 2 \cdot 75 + 2 \cdot 50 + 2 \cdot 25 + 2 \cdot 5 + 2 \cdot 0 = 310
\]

Since the velocity is apparently decreasing on the whole interval, we expect the LHS is an *overestimate* of the actual distance travelled and the RHS is an *underestimate* of the
actual distance. This says $310 \leq \text{actual distance} \leq 490$. But it is not possible to say for sure from this information whether a collision occurred.

B) (10) From the given information, what is your best estimate about whether the car hit the obstacle?

Solution: The average $(RHS + LHS)/2 = 400$ exactly. There is a good chance the car did hit or just brush the obstacle.

III.
A) (10) State the first and second parts of the Fundamental Theorem of Calculus.

Solution: See text or class notes.

B) (10) The following graph shows $y = f(x)$ for some function. Sketch the graph of the antiderivative $F(x)$ of $g(x)$ with $F(x) = 0$. What is $F(4)$?

Solution: The graph of $F$ should have a linear segment from $x = 0$ to $x = 1$ of slope 1, so $F(1) = 1$. Then between $x = 1$ and $x = 2$, the graph is part of a parabola opening down. The areas above and below the $x$-axis between $x = 1$ and $x = 2$ exactly balance out so $F(2) = F(1) = 1$. Between $x = 2$ and $x = 3$, the graph is part of a parabola opening up; the area below the axis is $-1/2$ (a triangle of base 1 and height 1). So $F(3) - F(2) = -1/2$ and $F(3) = 1/2$. Then the rest of the graph to $x = 4$ is part of a parabola opening down and $F(4) - F(3) = -1/2$, so $F(4) = 0$.

IV. Methods of integration. In B,C,D you may use the table of integrals.
A) (10) Using integration by parts, show that

$$\int x^n \cos(ax) \, dx = \frac{x^n}{a} \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax) \, dx.$$  

Solution: Let $u = x^n$, $dv = \cos(ax) \, dx$. Then $du = nx^{n-1}$ and $v = \frac{1}{a} \sin(ax)$, so using the integration by parts formula,

$$\int x^n \cos(ax) \, dx = \int u \, dv = uv - \int v \, du = \frac{x^n}{a} \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax) \, dx$$

which is what we had to show.

B) (10) $\int \tan(\sqrt{x})/\sqrt{x} \, dx$

Solution: (This is a $u$-substitution form.) Let $u = \sqrt{x} = x^{1/2}$. Then $du = \frac{1}{2\sqrt{x}}$ so the $\sqrt{x}$ in the denominator of the integrand is part of $du$. We have, using # 7 in the table (or by another substitution)

$$\int \tan(\sqrt{x})/\sqrt{x} \, dx = 2 \int \tan(u) \, du$$

$$= -2 \ln(\cos(u)) + C$$

$$= -2 \ln(\cos(\sqrt{x})) + C$$
C) \( \int (3x + 2)/(x^2 + 8x + 7) \, dx \)

*Solution:* The quadratic in the denominator factors as \( x^2 + 8x + 7 = (x + 1)(x + 7) \) (or, completing square, get \( (x + 4)^2 - 9 \) and the *negative sign* says there are real roots). Hence we want to use \#27 in the table with \( a = -1, b = -7, c = 3, d = 2 \). The result is

\[
\frac{1}{6}(((−1)(3) + 2)\ln |x + 1| - ((−7)(3) + 2)\ln |x + 7|) + C
\]

which simplifies to

\[
\frac{1}{6} (−\ln |x + 1| + 19 \ln |x + 7|) + C
\]

D) \( \int (4 - x^2)^{-3/2} \, dx \)

*Solution:* (This is a trigonometric substitution form.) Let \( x = 2\sin(\theta) \), so \( dx = 2\cos(\theta)\,d\theta \). Then the integral becomes

\[
\int (4 - x^2)^{-3/2} \, dx = 2 \int (4\cos^2(\theta))^{-3/2} \cos(\theta) \, d\theta
\]

\[
= \frac{1}{4} \int \cos^{-3}(\theta) \cos(\theta) \, d\theta
\]

\[
= \frac{1}{4} \int \cos^{-2}(\theta) \, d\theta
\]

\[
= \frac{1}{4} \int \sec^2(\theta) \, d\theta
\]

\[
= \frac{1}{4} \tan(\theta) + C
\]

To convert back to \( x \), we use \( \sin(\theta) = x/2 \), so \( \cos(\theta) = \sqrt{4 - x^2}/2 \), and \( \tan(\theta) = \sin(\theta)/\cos(\theta) = x/\sqrt{4 - x^2} \). The final answer is

\[
\int (4 - x^2)^{-3/2} \, dx = \frac{1}{4} \frac{x}{\sqrt{4 - x^2}} + C
\]