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ERRATUM

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Erratum to "The ubiquity of order domains for the construction of error control codes" (Advances in Mathematics of Communications, Vol.1, no.1, 2007, 151–171).

Let $\mathcal{X}$ be a projective variety defined over $\mathbb{F}_q$, and let v be a valuation on the function field $K(\mathcal{X})$. In Theorem 2 of [1], it was claimed that if v has rational rank $d = \dim \mathcal{X}$ and v is centered at a smooth point of $\mathcal{X}$ at infinity, then $\rho = -v|_R$ defines an order function on the affine coordinate ring R. The author would like to thank Matti Viikinkoski for pointing out that these hypotheses do not necessarily imply that $v(f) \leq 0$ for all $f \in R$, hence this claim is incorrect as stated.

For instance, there are examples of valuations on the function fields of surfaces such as the following (Example 3.4 of [2]). Let $\mathcal{X} = \mathbb{P}^2$ over $\mathbb{F}_q$, let Q be the point at infinity with homogeneous coordinates $(X : Y : Z) = (1 : 0 : 0)$, and let $u = Y/X$ and $w = Z/X$ be local coordinates at Q. The local ring $\mathcal{O}_{\mathcal{X},Q}$ of $\mathcal{X}$ at Q is isomorphic to $\mathbb{F}_q[u, w]_{(u,w)}$. We can define $v(w) = 1$ and $v(u) = \sqrt{2}$ on $\mathcal{O}_{\mathcal{X},Q}$ and extend v to the function field of $\mathcal{X}$. This yields a non-discrete valuation whose value group is isomorphic to the additive group of real numbers of the form $r + s\sqrt{2}$, $r, s \in \mathbb{Z}$, which contains elements arbitrarily close to zero.

The valuation ring S_v can be understood in terms of a sequence of blow-ups of $\mathcal{X}$, or in terms of the continued fraction expansion of the irrational number $\sqrt{2}$:

$$\sqrt{2} = [a_0; a_1, a_2, \cdots] = [1; 2, 2, \cdots].$$

The valuation ring S_v is generated by $u_0 = u$, $w_0 = w$ and the sequence of functions produced by iterating

$$w_{i+1} = u_i/w_i^{a_i}, \qquad u_{i+1} = w_i,$$

which yields

$$w_1 = \frac{u}{w}, \quad w_2 = \frac{w^3}{u^2}, \quad w_3 = \frac{u^5}{w^7}, \quad \cdots .$$

The valuations of this sequence, $v(w_1) = \sqrt{2} - 1$, $v(w_2) = 3 - 2\sqrt{2}$, $v(w_3) = 5\sqrt{2} - 7$ are all positive and $\lim_{n\to\infty} v(w_n) = 0$. Now, the odd-numbered terms in the sequence

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of w_i are actually *polynomials in the affine coordinates* $x = X/Z$ and $y = Y/Z$. For instance, $w_1 = \frac{u}{w} = \frac{Y/X}{Z/X} = y$,

$$w_3 = \frac{u^5}{v^7} = \frac{(Y/X)^5}{(Z/X)^7} = x^2y^5,$$

and so forth. Hence v is not negative on all of the elements of the affine coordinate ring $R = \mathbb{F}_q[x, y]$, even though every nonconstant element of R has a pole along the hyperplane at infinity. As a result, $\rho = -v|_R$ does not give an order function on R in this case. Not every non-discrete valuation gives counterexamples to Theorem 2 of [1], however. So the theorem should be corrected as follows.

Theorem 2. *Let R be an affine domain over $\mathbb{F}_q$, that is*

$$R \cong \mathbb{F}_q[X_1, \ldots, X_s]/I,$$

where I is a prime ideal. Let $\mathcal{X}$ be the projective closure of $V(I)$ in $\mathbb{P}^s$, and let H_0 (with reduced scheme structure) be the intersection of $\mathcal{X}$ with the hyperplane at infinity. Assume H_0 is an irreducible divisor on $\mathcal{X}$. Let v be any valuation on the function field $K(\mathcal{X})$ with value group Λ such that

1. *the rational rank of v is $d = \dim \mathcal{X}$, and*
2. *v is centered at a smooth point $Q \in H_0 \subset \mathcal{X}$.*
3. *$v(f) \leq 0$ in Λ for all $f \in R$.*

Then $\rho = -v|_R$ is an order function on R.

The proof given in [1] erroneously claims that condition (3) follows from the others. With the extra hypothesis, the proof goes through.

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