

Mathematics 136 – Calculus 2  
Summary of Trigonometric Substitutions  
February 12 and 14, 2020

The trigonometric substitution method handles many integrals containing expressions like

$$\sqrt{a^2 - x^2}, \sqrt{x^2 + a^2}, \sqrt{x^2 - a^2}$$

(possibly including expressions without the square roots!) The basis for this approach is the trigonometric identities

$$\begin{aligned} 1 &= \sin^2 \theta + \cos^2 \theta \\ \Rightarrow \sec^2 \theta &= \tan^2 \theta + 1. \end{aligned}$$

from which we derive other related identities:

$$\begin{aligned} \sqrt{a^2 - (a \sin \theta)^2} &= \sqrt{a^2(1 - \sin^2 \theta)} = \sqrt{a^2 \cos^2 \theta} = a \cos \theta \\ \sqrt{(a \tan \theta)^2 + a^2} &= \sqrt{a^2(\tan^2 \theta + 1)} = \sqrt{a^2 \sec^2 \theta} = a \sec \theta \\ \sqrt{(a \sec \theta)^2 - a^2} &= \sqrt{a^2(\sec^2 \theta - 1)} = \sqrt{a^2 \tan^2 \theta} = a \tan \theta \end{aligned}$$

(Technical note: We usually assume that  $a > 0$  and  $0 < \theta < \pi/2$  here so that all the trig functions take positive values.) Hence,

1. If our integral contains  $\sqrt{a^2 - x^2}$ , the substitution  $x = a \sin \theta$  will convert this radical to the simpler form  $a \cos \theta$ .
2. If our integral contains  $\sqrt{x^2 + a^2}$ , the substitution  $x = a \tan \theta$  will convert this radical to the simpler form  $a \sec \theta$ .
3. If our integral contains  $\sqrt{x^2 - a^2}$ , the substitution  $x = a \sec \theta$  will convert this radical to the simpler form  $a \tan \theta$ .

We substitute for the rest of the integral *including the dx*. For instance if  $x = a \sin \theta$ , then the  $dx = a \cos \theta d\theta$ . If  $x = a \tan \theta$ , then  $dx = a \sec^2 \theta d\theta$ . If  $x = a \sec \theta$ , then  $dx = a \sec \theta \tan \theta d\theta$ . All these substitutions should produce an integral with no  $x$  terms left – everything expressed in terms of  $\theta$ . If desired, limits of integration on definite integrals can also be converted to equivalent  $\theta$ -values.

We then integrate the resulting trigonometric form using the trig methods discussed last time, and convert back to the original variable (or substitute limits of integration in terms of  $\theta$ ).

### Two Examples

A) Compute  $\int \frac{u^2}{\sqrt{a^2 - u^2}} du$ . Solution: The  $\sqrt{a^2 - u^2}$  tells us that we want the sine substitution:  $u = a \sin \theta$ . Then  $du = a \cos \theta d\theta$ , and the integral becomes:

$$\int \frac{a^3 \sin^2 \theta \cos \theta d\theta}{a \cos \theta} = a^2 \int \sin^2 \theta d\theta$$

We can apply one of our trigonometric reduction formulas to yield

$$\frac{a^2\theta}{2} - \frac{a^2}{2} \sin\theta \cos\theta + C.$$

Alternatively, a trigonometric identity (the half-angle formula  $\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$ ) can be applied to this:

$$a^2 \int \frac{1}{2}(1 - \cos(2\theta)) d\theta$$

then we integrate to yield

$$\begin{aligned} &= \frac{a^2}{2}\theta - \frac{a^2}{4} \sin(2\theta) \\ &= \frac{a^2\theta}{2} - \frac{a^2}{2} \sin\theta \cos\theta + C. \end{aligned}$$

Note we used the double-angle formula for sin to get from the first line to the second. Then, we convert back to functions of  $u$  using the substitution equation  $u = a \sin\theta$ . From this,

$$\theta = \arcsin(u/a), \quad \cos\theta = \sqrt{a^2 - u^2}/a, \quad \sin\theta = u/a$$

so the integral is

$$\int \frac{u^2}{\sqrt{a^2 - u^2}} du = \frac{a^2}{2} \arcsin(u/a) - \frac{1}{2}u\sqrt{a^2 - u^2} + C.$$

B) Compute  $\int \frac{dx}{\sqrt{x^2+16}}$ . Solution: The  $\sqrt{x^2+16}$  indicates that we want the tangent substitution  $x = 4 \tan\theta$ . Then  $dx = 4 \sec^2\theta d\theta$  and the integral becomes:

$$\int \frac{4 \sec^2\theta d\theta}{4 \sec\theta} = \int \sec\theta d\theta.$$

As we saw last time, this form can be integrated as follows:

$$\ln|\sec\theta + \tan\theta| + C$$

Then, from  $x = 4 \tan\theta$ , we get  $\sec\theta = \frac{\sqrt{x^2+16}}{4}$  and  $\tan\theta = \frac{x}{4}$ . Hence the integral equals:

$$\ln \left| \frac{\sqrt{x^2+16}}{x} + \frac{4}{x} \right| + C.$$

Note: If this was a definite integral, say

$$\int_0^4 \frac{dx}{\sqrt{x^2+16}},$$

then the equivalent limits of integration in terms of  $\theta$  would be found from the formula  $\theta = \tan^{-1}(x/4)$ , so  $x = 0 \Leftrightarrow \theta = 0$  and  $x = 4 \Leftrightarrow \theta = \pi/4$ . If we used these values, then we could apply the Evaluation Theorem to the trig integral, and it would not be necessary to convert back to the antiderivative in terms of  $x$ .