MATH 136 – Calculus 2
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Background If a solid extends along the x-axis between a and b and the area of the cross-section by a plane $x =$ const is given by some function $A(x)$ for all x, then *Cavalieri's Principle* (named after the Italian Renaissance mathematician Bonaventura Cavalieri, 1598-1647) says that

$$\text{Volume} = \int_a^b A(x)\, dx$$

Questions

1. A circular cone extends from $x = 0$ to $x = 4$ along the x-axis. The cross-section in each plane $x =$ constant is a circle whose radius increases linearly from $r = 0$ at $x = 0$ to $r = 5$ by the time x reaches 4.

   (a) Write the radius r as a function of x on the interval 0 to 4.

   Solution: The radius is

   $$r(x) = 5x/4$$

   (b) Write the area of the circular cross-section as a function of x.

   Solution: $A(x) = \pi(5x/4)^2 = \frac{25\pi x^2}{16}$.

   (c) Find the volume by applying the Cavalieri Principle equation above.

   Solution: The volume is

   $$\int_0^4 \frac{25\pi x^2}{16}\, dx = \left.\frac{25\pi x^3}{48}\right|_0^4 = \frac{100\pi}{3}.$$

   (d) Check your result with the formula for the volume of a cone from high school geometry.

   Solution: The formula is $V_{\text{cone}} = \frac{1}{3}\pi R^2 h$ where R is the radius of the base, and h is the height. Here $R = 5$ and $h = 4$, so $V_{\text{cone}} = \frac{100\pi}{3}$, which agrees with the value computed above(!)

2. A solid extends along the x-axis from $x = -3$ to $x = 3$. The cross-section in each plane $x =$ constant is a semicircle with radius e^x. Find the volume of the solid using Cavalieri's Principle.

Solution: The volume is

$$V = \int_{-3}^{3} \frac{\pi}{2}(e^x)^2 \, dx = \int_{-3}^{3} \frac{\pi}{2} e^{2x} \, dx = \frac{\pi}{4} e^{2x} \Big|_{-3}^{3} = \frac{\pi}{4}(e^6 - e^{-6}).$$

3. A solid extends from $x = 0$ to $x = 10$ along the x-axis. The cross-section at x is a square with side $\sqrt[4]{25 + x^2}$. Set up the integral to compute the volume using Cavalieri's Principle. Evaluate the integral using a trigonometric substitution or a table of integrals.

Solution: The volume is

$$\int_{0}^{10} (\sqrt[4]{x^2 + 25})^2 \, dx = \int_{0}^{10} \sqrt{x^2 + 25} \, dx.$$

Using the trig subsitution $x = 5 \tan \theta$ or a table,

$$= \frac{x\sqrt{x^2 + 25}}{2} + \frac{25}{2} \ln |x + \sqrt{x^2 + 25}| \Big|_{0}^{10} \doteq 73.95.$$

4. In the previous problem, would it make an difference if the squares all had center point along the x-axis or if all the squares had one corner at a point on the x-axis?

Solution: No, the volumes would be equal(!)