MATH 136 – Calculus 2
Practice Day on Trigonometric Substitution Integrals
February 14, 2020

Background

Recall that we saw some first examples of the *trigonometric substitution* method last time. The basic outline of this method is that for integrals involving

- $\sqrt{a^2 - x^2}$, we let $x = a\sin\theta$ and $dx = a\cos\theta\ d\theta$
- $\sqrt{a^2 + x^2}$, we let $x = a\tan\theta$ and $dx = a\sec^2\theta\ d\theta$
- $\sqrt{x^2 - a^2}$, we let $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta\ d\theta$

We simplify, then apply our trig reduction formulas from last week. The last step is to convert back to functions of x using the reference triangle corresponding to the substition used:

- For the $x = a\sin\theta$ substitution, put x on opposite side, a on hypotenuse, then $(adj) = \sqrt{a^2 - x^2}$, so you can read off any trig function of θ from the triangle and $\theta = \sin^{-1}(x/a)$
- For the $x = a\tan\theta$ substitution, put x on opposite side, a on adjacent, then $(hyp) = \sqrt{x^2 + a^2}$, so you can read off any trig function of θ from the triangle and $\theta = \tan^{-1}(x/a)$
- For the $x = a\sec\theta$ substitution, put x on hypotenuse, a on adjacent, then $(opp) = \sqrt{x^2 - a^2}$, so you can read off any trig function of θ from the triangle and $\theta = \sec^{-1}(x/a)$.

Questions

(A) Using the appropriate trigonometric substitution compute

$$\int \frac{1}{\sqrt{x^2 + 25}}\, dx$$

Answer: Let $x = 5\tan(\theta)$ and $dx = 5\sec^2(\theta)\ d\theta$. The integral becomes

$$\begin{aligned} \int \frac{5\sec^2(\theta)\ d\theta)}{5\sec(\theta)} &= \int \sec(\theta)\ d\theta \\ &= \ln|\sec(\theta) + \tan(\theta)| + C \\ &= \ln|\sqrt{x^2+25} + x| + C \end{aligned}$$

Note: You could also write this as

$$\ln\left|\frac{\sqrt{x^2+25}}{5} + \frac{x}{5}\right| + C$$

but by properties of logarithms, the $-\ln(5)$ from the denominator can be absorbed into the constant of integration, so either form is correct.

(B) Using the appropriate trigonometric substitution compute

$$\int \frac{1}{(36-x^2)^{3/2}}\ dx$$

Answer: Let $x = 6\sin(\theta)$ and $dx = 6\cos(\theta)\ d\theta$. The integral becomes

$$\frac{1}{36}\int \cos^{-2}(\theta)\ d\theta = \frac{1}{36}\int \sec^2(\theta)\ d\theta = \frac{1}{36}\tan(\theta) + C.$$

After setting up the triangle, we see $\sin(\theta) = \frac{x}{6}$ is the opposite over hypotenuse. The adjacent side is $\sqrt{36-x^2}$ and the tangent is opposite over adjacent or

$$= \frac{x}{36\sqrt{36-x^2}} + C.$$

(C) Using the appropriate trigonometric substitution compute

$$\int \frac{1}{\sqrt{x^2-16}}\ dx$$

Answer: The form $x^2 - 16$ under the square root calls for the substitution $x = 4\sec(\theta)$, so $dx = 4\sec(\theta)\tan(\theta)\ d\theta$. The integral becomes

$$\int \frac{4\sec(\theta)\tan(\theta)\ d\theta}{4\tan(\theta)} = \int \sec(\theta)\ d\theta = \ln|\sec(\theta) + \tan(\theta)| + C.$$

To convert back to functions of x, from $\sec(\theta) = \frac{x}{4}$, we set up a triangle containing the angle θ with hypotenuse x and adjacent side 4. The opposite side is $\sqrt{x^2 - 16}$. Then $\sec(\theta) = \frac{x}{4}$ and $\tan(\theta) = \frac{\sqrt{x^2-16}}{4}$. As in the integral from (A), the denominators can be absorbed into the constant of integration and the answer can be written as

$$= \ln|x + \sqrt{x^2 - 16}| + C$$

(D) You want to divide a circular pizza with radius 9in (say with outer crust along the circle $x^2 + y^2 = 81$) into three exactly equal pieces with a knife or pizza cutter. One way, of course, is to divide it into three sectors each with angle exactly $2\pi/3$ by cutting from the center to the outer edge. But another (easier?) way would be to cut the pizza with two parallel strokes, say at $x = \pm a$ so that the three strips $-9 \le x \le -a$, $-a \le x \le a$, and $a \le x \le 9$ all have the same area. Set up an integral with a in the limits of integration, evaluate it using an appropriate trigonometric substitution, and find an equation to solve for the a giving three strips of exactly equal area. Find an approximation for a by using a graphing calculator.

Answer: By the circular symmetry, we only need to find a to make:

$$\int_{-a}^{a} \sqrt{81 - x^2}\, dx = \int_{a}^{9} \sqrt{81 - x^2}\, dx$$

The integrals can be computed using the FTC, part I and the antiderivative formula

$$\int \sqrt{81 - x^2}\, dx = \frac{x\sqrt{81 - x^2}}{2} + \frac{81 \sin^{-1}(x/9)}{2}$$

(from the trig substition $x = 9\sin(\theta)$). We get an equation:

$$\frac{3a}{2}\sqrt{81 - a^2} + \frac{243\, \sin^{-1}(a/9)}{2} - \frac{81\,\pi}{4} = 0$$

If you plot this function of a, you can see that the curve crosses the horizonal axis at about $a \doteq 2.384$. Slicing the pie at $x = \pm a = \pm 2.384$ will yield three pieces of equal areas (but different shapes of course!)