MATH 136 – Calculus 2
An Application of Integration By Parts:
Reduction Formulas
February 10, 2020

Background

There are a number of cases where repeated integration by parts can be used to completely evaluate integrals. Instead of working these out laboriously by hand every time, many mathematicians who need to use these will use a *reduction formula* that shows the general pattern of each application of the parts formula. These are usually obtained by consulting a compiled *table of integrals* rather than worked out by hand. Here is an example:

$$\int x^n e^{ax}\ dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\ dx$$

(This is derived by letting $u = x^n$ and $dv = e^{ax}\ dx$ as in an example from the video on "parts.") If we want to integrate $\int x^3 e^{4x}\ dx$, for instance, we apply this reduction formula three times with $n = 3$, then $n = 2$, then $n = 1$. The power of x is eventually reduced to a constant 1.

$$\begin{aligned}\int x^3 e^{4x}\ dx &= \frac{x^3 e^{4x}}{4} - \frac{3}{4}\int x^2 e^{4x}\ dx \\ &= \frac{x^3 e^{4x}}{4} - \frac{3}{4}\left(\frac{x^2 e^{4x}}{4} - \frac{2}{4}\int x e^{4x}\ dx\right) \\ &= \frac{x^3 e^{4x}}{4} - \frac{3x^2 e^{4x}}{16} + \frac{3}{8}\left(\frac{x e^{4x}}{4} - \frac{1}{4}\int e^{4x}\ dx\right) \\ &= \frac{x^3 e^{4x}}{4} - \frac{3x^2 e^{4x}}{16} + \frac{3x e^{4x}}{32} - \frac{3e^{4x}}{128} + C.\end{aligned}$$

Questions

(A) Derive these reduction formulas for $n \geq 1$:

$$\int x^n \cos(ax)\ dx = \frac{x^n \sin(ax)}{a} - \frac{n}{a}\int x^{n-1}\sin(ax)\ dx$$

and

$$\int x^n \sin(ax)\ dx = \frac{-x^n \cos(ax)}{a} + \frac{n}{a}\int x^{n-1}\cos(ax)\ dx$$

by integrating by parts (once each).

Answer: Let $u = x^n$ and $dv = \cos(ax)\ dx$ for the first and $dv = \sin(ax)\ dx$ for the second. The formula follows immediately from the parts formula since $du = nx^{n-1}\ dx$ and $v = \frac{\sin(ax)}{a}$ for the first and $v = \frac{-\cos(ax)}{a}$ for the second.

(B) Using the two reduction formulas from part (A) in sequence, integrate:

$$\int x^2 \cos(3x)\ dx$$

Answer: We use the cosine formula first with $n = 2$, then the sine formula with $n = 1$:

$$\begin{aligned}\int x^2 \cos(3x)\ dx &= \frac{x^2 \sin(3x)}{3} - \frac{2}{3} \int x \sin(3x)\ dx \\ &= \frac{x^2 \sin(3x)}{3} - \frac{2}{3} \left(\frac{-x\cos(3x)}{3} + \frac{1}{3} \int \cos(3x)\ dx \right) \\ &= \frac{x^2 \sin(3x)}{3} + \frac{2x\cos(3x)}{9} - \frac{2\sin(3x)}{27} + C.\end{aligned}$$

(C) Derive the following reduction formula by applying parts with $u = \sin^{n-1}(x)$ and $dv = \sin(x)\ dx$:

$$\int \sin^n(x)\ dx = \frac{-\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n} \int \sin^{n-2}(x)\ dx.$$

Answer: We have $du = (n-1)\sin^{n-2}(x)\cos(x)\ dx$ and $v = -\cos(x)$. Applying the parts formula, we have

$$\int \sin^n(x)\ dx = -\sin^{n-1}(x)\cos(x) + (n-1) \int \sin^{n-2}(x)\cos^2(x)\ dx$$

Now, we use the trig identity $\cos^2(x) = 1 - \sin^2(x)$ to subsitute for the factor $\cos^2(x)$ in the integral on the right, multiply out, and split into two integrals. This yields the equation:

$$\int \sin^n(x)\ dx = -\sin^{n-1}(x)\cos(x) + (n-1) \int \sin^{n-2}(x)\ dx - (n-1) \int \sin^n(x)\ dx$$

The last term on the right is the same integral we started with. But since we have the coefficient $-(n-1)$ we can add $(n-1)\int \sin^n(x)\ dx$ to both sides then divide by n to get the required formula.

(D) Use the formula from (C) to compute $\int \sin^5(x)\,dx$.

Answer: We use the formula with $n = 5$ and $n = 3$. When we get down to the first power, we don't need the formula any more!

$$\begin{aligned}\int \sin^5(x)\,dx &= \frac{-\sin^4(x)\cos(x)}{5} + \frac{4}{5}\int \sin^3(x)\,dx \\ &= \frac{-\sin^4(x)\cos(x)}{5} + \frac{4}{5}\left(\frac{-\sin^2(x)\cos(x)}{3} + \frac{2}{3}\int \sin(x)\,dx\right) \\ &= \frac{-\sin^4(x)\cos(x)}{5} - \frac{4\sin^2(x)\cos(x)}{15} - \frac{8\cos(x)}{15} + C\end{aligned}$$

(Note: The trickiest part of these is remembering to multiply all the factors that are introduced by the $\frac{n-1}{n}$ multiplying the integral each time you apply the formula!)

(E) Derive the following reduction formula for $n \geq 2$:

$$\int \tan^n(u)\,du = \frac{\tan^{n-1}(u)}{n-1} - \int \tan^{n-2}(u)\,du.$$

(Hint: This does *not* come from an integral by parts, but rather uses a "clever" application of the trig identity $\tan^2(x) = \sec^2(x) - 1$.) The case $n = 1$ is handled by a separate formula we have seen before (u-subsitution):

$$\int \tan(u)\,du = -\ln|\cos(u)| + C$$

Answer: We break the $\tan^n(x)$ up as $\tan^{n-2}(x)\cdot\tan^2(x)$ then use the identity. Separating into two integrals gives:

$$\int \tan^n(x)\,dx = \int \tan^{n-2}(x)\sec^2(x)\,dx - \int \tan^{n-2}(x)\,dx.$$

The first integral has the form $\int u^{n-2}\,du$ for $u = \tan(x)$. This yields

$$\int \tan^n(x)\,dx = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)\,dx$$

and we're done! Note again: This is *not integration by parts!*

(F) Use the formulas from (E) to integrate

$$\int \tan^5(x)\ dx$$

Answer: We have

$$\begin{aligned} \int \tan^5(x)\ dx &= \frac{\tan^4(x)}{4} - \int \tan^3(x)\ dx \\ &= \frac{\tan^4(x)}{4} - \frac{\tan^2(x)}{2} + \int \tan(x)\ dx \\ &= \frac{\tan^4(x)}{4} - \frac{\tan^2(x)}{2} - \ln|\cos(x)| + C. \end{aligned}$$