MATH 136 – Calculus 2
Practice Day on Consequences of FTC, "Part II"
January 29, 2020

Background

Recall that the "FTC, part II" says: If f is continuous on $[a, b]$, then

$$\frac{d}{dx} \int_a^x f(t)\ dt = f(x)$$

for all x in $[a, b]$. (Note: at the endpoints $x = a$ and $x = b$, we may be talking only about "one-sided derivatives" if f is not defined for $x < a$ or $x > b$.)

Questions

(0) Look at the function $f(t)$ plotted on the back. What is $\frac{d}{dx} \int_0^x f(t)\ dt$ at $x = 3$?

Answer: It is $f(3) = 2$.

(1) What is

$$\frac{d}{dx} \int_3^x \sqrt[3]{t^5 + 3t + 1}\ dt?$$

Answer: This is $\sqrt[3]{x^5 + 3x + 1}$.

(2) Same question for

$$\frac{d}{dx} \int_0^x \frac{\tan(t)}{t^3 + 2}\ dt$$

Answer: This is $\frac{\tan(x)}{x^3+2}$.

(3) What is

$$\frac{d}{dx} \int_x^4 e^{4t^2+7}\ dt?$$

Answer: Since the x is in the lower limit, we need to reverse the limits of integration, which introduces a negative sign, then apply "FTC, part II"

$$\begin{aligned} &= -\frac{d}{dx} \int_4^x e^{4t^2+7}\ dt \\ &= -e^{4x^2+7}. \end{aligned}$$

the derivative is $-e^{4x^2+7}$.

(4) What about

$$\frac{d}{dx}\int_0^{x^2}\sqrt{t^2+1}\,dt?$$

Answer: Now we need the FTC, part II, and the chain rule: To see this note that we are looking at $F(x^2)$ where $F(x)=\int_0^x\sqrt{t^2+1}\,dt$. Then by the chain rule, $\frac{d}{dx}F(x^2)=F'(x^2)\cdot 2x$. The FTC, part II gives $F'(x^2)=\sqrt{x^4+1}$ and $2x$ is the "derivative of the inside x^2." So the final answer is $2x\sqrt{x^4+1}$.

(5) What is

$$\frac{d}{dx}\int_{-x^3}^{x^4}\frac{t}{t+1}\,dt$$

(You'll need to use the interval union property, the interchange of limits property and the chain rule here!)

Answer: Using the hints:

$$\begin{aligned}\frac{d}{dx}\int_{-x^3}^{x^4}\frac{t}{t+1}\,dt &= \frac{d}{dx}\left(\int_{-x^3}^{0}\frac{t}{t+1}\,dt+\int_0^{x^4}\frac{t}{t+1}\,dt\right)\\ &= \frac{d}{dx}\left(-\int_0^{-x^3}\frac{t}{t+1}\,dt+\int_0^{x^4}\frac{t}{t+1}\,dt\right)\\ &= \frac{-x^3}{-x^3+1}\cdot(-3x^2)+\frac{x^4}{x^4+1}\cdot 4x^3.\end{aligned}$$

(6) Finally, suppose someone asked you for a function whose derivative was $f(x)=\dfrac{x^2}{x^4+x+1}$. How could you "cook one up" using what we have talked about? (In case you are worried about possible discontinuities from vertical asymptotes, you don't need to: this function f is continuous for all real x – the denominator is never zero.)

Answer: The FTC, part II says that

$$F(x)=\int_0^x\frac{t^2}{t^4+t+1}\,dt$$

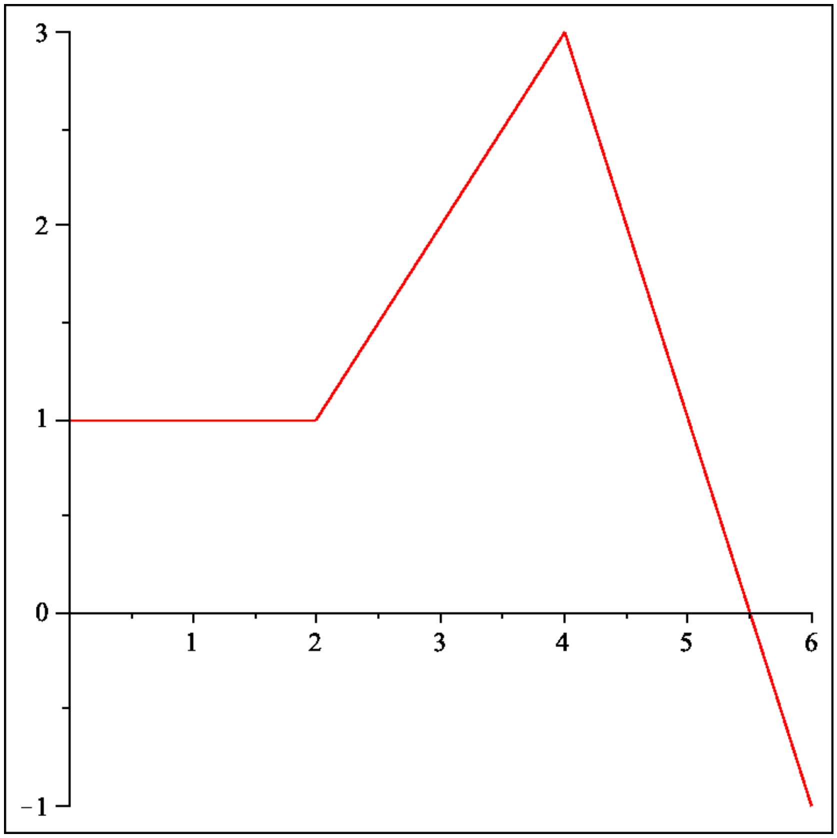

Figure 1: The region

is one such function. Starting from any lower limit of integration a rather than zero gives others.