4, 2, 4 total 10



MATH 136 Problem Set 4, Part B Solutions

§ 7.5 / 36 $\int \frac{dx}{x^2(x^2+25)}$

Partial fractions:

$$\frac{1}{x^2(x^2+25)} = \frac{A}{x^2} + \frac{B}{x} + \frac{Cx+D}{x^2+25}$$ 1

So
$$1 = A(x^2+25) + Bx(x^2+25) + (Cx+D)x^2$$
$$= (B+C)x^3 + (A+D)x^2 + 25Bx + 25A$$

So $B+C=0,\quad A+D=0 \quad 25B=0 \quad 25A=1$ 1

$\Rightarrow A = \frac{1}{25},\ B=0,\ C=0,\ D = -\frac{1}{25}$

$$\int \frac{dx}{x^2(x^2+25)} = \int \frac{1/25}{x^2}\,dx + \int \frac{-1/25}{x^2+25}\,dx$$

$$= \boxed{-\frac{1}{25x} - \frac{1}{125}\tan^{-1}\left(\frac{x}{5}\right) + C}$$

1 1

Note: we have seen the general formula $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$ so it's OK if they just quote that, or they might rewrite $\int \frac{1}{x^2+25}\,dx$

$= \frac{1}{25}\int \frac{1}{\left(\frac{x}{5}\right)^2+1}\,dx$ + use a u-sub.

or they might use the trig. sub $x = 5\tan\theta$

All are OK.

§ 7.6/48

$$\int e^x \sqrt{e^{2x}-1}\,dx \qquad \text{Let } \overbrace{u = e^x \quad du = e^x\,dx}^{1}$$

$$= \int \sqrt{u^2-1}\,du \qquad \text{Let } u = \sec\theta \quad du = \sec\theta\tan\theta\,d\theta$$

Also OK to use #76 in book's table of integrals here directly

$$= \int \sqrt{\sec^2\theta - 1}\,\sec\theta\tan\theta\,d\theta$$

$$= \int \sec\theta\tan^2\theta\,d\theta$$

$$= \int \sec\theta(\sec^2\theta - 1)\,d\theta$$

$$= \int \sec^3\theta - \sec\theta\,d\theta$$

using reduction formula or a table

$$= \frac{\sec\theta\tan\theta}{2} - \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

#13, #42 in book's table

$$= \frac{u\sqrt{u^2-1}}{2} - \frac{1}{2}\ln|u + \sqrt{u-1}| + C$$

$$= \boxed{\frac{e^x\sqrt{e^{2x}-1}}{2} - \frac{1}{2}\ln|e^x + \sqrt{e^{2x}-1}| + C} \quad 1$$

(1 point for $u = e^x$ to start then 1 point for final answer)

§ 7.7 | 101

(b) $\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}\,dt$, $n \geq 1$.

$$= \lim_{b\to\infty} \int_0^b t^{n-1}e^{-t}\,dt$$

$$\text{so } \Gamma(n+1) = \lim_{b\to\infty} \int_0^b t^n e^{-t}\,dt$$

Let $u = t^n$ $dv = e^{-t}\,dt$, so $du = nt^{n-1}\,dt$, $v = -e^{-t}$ 1 so $\Gamma(n+1) = \lim_{b\to\infty}\left[ -t^n e^{-t} \Big|_0^b + n\int_0^b t^{n-1} e^{-t}\,dt \right]$

$= \lim_{b\to\infty} \frac{-t^n}{e^t} + \lim_{b\to\infty} n \cdot \int_0^b t^{n-1} e^{-t}\,dt$

$= 0 + n\Gamma(n)$ 1 using L'Hopital's Rule

$\therefore \boxed{\Gamma(n+1) = n\Gamma(n)}$

(c) $\Gamma(1) = \int_0^\infty e^{-t}\,dt$

$= \lim_{b\to\infty} -e^{-t}\Big|_0^b$

$= \lim_{b\to\infty} 1 - e^{-b}$

$= 1 = 0!$ 1

then using (b), $\Gamma(2) = 1\cdot\Gamma(1) = 1 = 1!$

$\Gamma(3) = 2\Gamma(2) = 2 = 2!$

$\Gamma(4) = 3\Gamma(2) = 3\cdot 2! = 3!$ 1

$\vdots$

$\Gamma(n+1) = n\Gamma(n) = n\cdot(n-1)! = n!$