MATH 136 – Problem Set 3, Part B Solutions

§7.1

76. Let $w = \ln x$ so $x = e^w$. Then $dw = \frac{1}{x}\,dx$ and $dx = x\,dw = e^w\,dw$. We get

$$\int \frac{(\ln x)^2}{x^2}\,dx = \int \frac{w^2}{e^{2w}} \cdot e^w\,dw$$

$$= \int w^2 e^{-w}\,dw$$

Now we integrate by parts twice with $u =$ power of w each time:

$$\int \underbrace{w^2}_{u} \overbrace{e^{-w}\,dw}^{dv} = -w^2 e^{-w} + 2\int \underbrace{w}_{u} \overbrace{e^{-w}\,dw}^{dv}$$

$$= -w^2 e^{-w} - 2we^{-w} + 2\int e^{-w}\,dw$$

$$= \boxed{-w^2 e^{-w} - 2we^{-w} - 2e^{-w}} + C.$$

Finally converting back to x:

$$\int \frac{(\ln(x))^2}{x^2}\,dx = \boxed{-\frac{(\ln x)^2}{x} - \frac{2\ln(x)}{x} - \frac{2}{x} + C}$$

82. Integrate by parts with $u = (\ln(x))^k$, $dv = dx$. Then $du = k(\ln(x))^{k-1} \cdot \frac{1}{x}\,dx$ (chain rule) and $v = x$, so

$$\int (\ln(x))^k\,dx = x(\ln(x))^k - \int k(\ln(x))^{k-1} \cdot \frac{1}{\cancel{x}} \cdot \cancel{x}\,dx$$

$$= x(\ln(x))^k - k\int (\ln(x))^{k-1}\,dx.$$

§7.2 / 7. To count the area below the x-axis as positive, we want

$$\int_0^{3\pi/2} |\cos^3 x|\, dx = \int_0^{\pi/2} \cos^3 x\, dx - \int_{\pi/2}^{3\pi/2} \cos^3 x\, dx$$

using the SC2 reduction formula

$$= \left.\frac{\cos^2(x)\sin(x)}{3} + \frac{2\sin x}{3}\right|_0^{\pi/2} - \left.\left(\frac{\cos^2(x)\sin(x)}{3} + \frac{2\sin(x)}{3}\right)\right|_{\pi/2}^{3\pi/2}$$

$$= \left(0 + \frac{2}{3}\right) - (0 + 0) - \left(0 - \frac{2}{3}\right) + \left(0 + \frac{2}{3}\right)$$

$$= \boxed{2} \checkmark$$

§7.3 / 30.

For $\int_0^1 \frac{dx}{(x^2+1)^3}$, consider $\int \frac{dx}{(x^2+1)^3}$ as $\int \frac{dx}{(\sqrt{x^2+1})^6}$

then letting $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$ and $\sqrt{x^2+1} = \sec\theta$, so

$$\int \frac{dx}{(\sqrt{x^2+1})^6} = \int \frac{\sec^2\theta\, d\theta}{\sec^6\theta\, d\theta}$$

$$= \int \cos^4\theta\, d\theta \quad \text{use SC2}$$

$$= \frac{\cos^3\theta\sin\theta}{4} + \frac{3}{4}\int \cos^2\theta \quad \text{use SC2 again}$$

$$= \frac{\cos^3\theta\sin\theta}{4} + \frac{3}{4}\left(\frac{\cos\theta\sin\theta}{2} + \frac{\theta}{2}\right)$$

$$= \frac{\cos^3\theta\sin\theta}{4} + \frac{3\cos\theta\sin\theta}{8} + \frac{3\theta}{8}$$

We save some work if we also convert the limits of integration: $x = 1 \Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$

$x = 0 \Rightarrow \theta = \tan^{-1}(0) = 0$

then

$$\int_0^1 \frac{dx}{(x^2+1)^3} = \int_0^{\pi/4} \cos^4\theta \, d\theta$$

$$= \frac{\cos^3\theta \sin\theta}{4} + \frac{3\cos\theta\sin\theta}{8} + \frac{3\theta}{8} \Big|_0^{\pi/4}$$

$$= \frac{\left(\frac{\sqrt{2}}{2}\right)^3 \cdot \frac{\sqrt{2}}{2}}{4} + \frac{3\,\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}}{8} + \frac{3\cdot\frac{\pi}{4}}{8} - 0$$

$$= \frac{1}{16} + \frac{3}{16} + \frac{3\pi}{32}$$

$$= \boxed{\frac{1}{4} + \frac{3\pi}{32}} \checkmark$$