MATH 136 – Calculus 2
First Practice on Improper Integrals
February 25, 2020

*Background*

Whenever $a = -\infty$ or $b = +\infty$ or both, the integral

$$\int_a^b f(x)\ dx$$

is said to be an *improper integral*. Improper integrals are always handled by *taking limits of "ordinary" integrals.*

- We say $\int_a^\infty f(x)\ dx$ *converges* if the limit

  $$\lim_{b\to\infty} \int_a^b f(x)\ dx$$

  exists and we say the integral *diverges* otherwise.

- We say $\int_{-\infty}^b f(x)\ dx$ *converges* if the limit

  $$\lim_{a\to-\infty} \int_a^b f(x)\ dx$$

  exists and we say the integral *diverges* otherwise.

- We say $\int_{-\infty}^\infty f(x)\ dx$ *converges* if, splitting at any finite value $c$ (often at $c = 0$), *both of the limits* here:

  $$\lim_{a\to-\infty} \int_a^c f(x)\ dx + \lim_{b\to+\infty} \int_c^b f(x)\ dx$$

  exist and we say the integral *diverges* otherwise. (Note this says that if either one of the two limits does not exist, then the integral diverges.)

When $[a, b]$ is a finite interval, the integral

$$\int_a^b f(x)\ dx$$

is also said to be *improper* whenever $f(x)$ has one or more *discontinuities* in the interval $[a, b]$. These can happen at the endpoints $a$ or $b$, or at some $c$ in $(a, b)$. In all cases, improper integrals are handled by *taking limits of "ordinary" integrals*. For the purposes of this discussion, let us suppose that $f(x)$ has only one point of discontinuity in the interval.

- If $f$ is discontinuous at $a$, then we say $\int_a^b f(x)\, dx$ *converges* if the limit

  $$\lim_{c \to a^+} \int_c^b f(x)\, dx$$

  exists and we say the integral *diverges* otherwise.

- If $f$ is discontinuous at $b$, then we say $\int_a^b f(x)\, dx$ *converges* if the limit

  $$\lim_{c \to b^-} \int_a^c f(x)\, dx$$

  exists and we say the integral *diverges* otherwise.

- If $f$ is discontinuous at $d$ in $(a, b)$, then we say $\int_a^b f(x)\, dx$ *converges* if *both of the limits* here:

  $$\lim_{c \to d^-} \int_a^c f(x)\, dx + \lim_{c \to d^+} \int_c^b f(x)\, dx$$

  exist and we say the integral *diverges* otherwise. (Note this says that if either one of the two limits does not exist, then the integral diverges.)

*Questions*

For each of the following,

(i) Explain why the integral is improper.

(ii) Set up the appropriate limit(s) and evaluate.

(iii) Say whether the integral converges or diverges.

1. $\displaystyle\int_{-\infty}^{2} e^{10x}\, dx$

*Solution:* This is improper because of the infinite lower limit of integration. We have

$$\lim_{a\to-\infty}\int_a^2 e^{10x}\,dx = \lim_{a\to-\infty}\frac{1}{10}(e^{20}-e^a) = \frac{1}{10}e^{20}.$$

Hence the integral *converges.*

2. $\displaystyle\int_4^\infty \frac{1}{x^2+16}\,dx$

   *Solution:* For this and the next example, we use the indefinite integration formula.

$$\int \frac{1}{x^2+16}\,dx = \frac{1}{4}\tan^{-1}(x/4)+C$$

   This integral is improper because of the infinite upper limit of integration. We have

$$\begin{aligned}\lim_{b\to\infty}\frac{1}{4}\tan^{-1}(x/4)\Big|_4^b &= \lim_{b\to\infty}\frac{1}{4}\left(\tan^{-1}(b/4)-\frac{\pi}{4}\right)\\ &= \frac{1}{4}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)\\ &= \frac{\pi}{16}.\end{aligned}$$

   The integral *converges.*

3. $\displaystyle\int_{-\infty}^\infty \frac{1}{x^2+16}\,dx$

   *Solution:* This is improper because both limits of integration are infinite. The integral equals

$$\lim_{a\to-\infty}\int_a^0 \frac{1}{x^2+16}\,dx + \lim_{b\to\infty}\int_0^b \frac{1}{x^2+16}\,dx.$$

   Using the formula from the previous problem, this equals

$$\lim_{a\to-\infty} -\frac{1}{4}\tan^{-1}(a/4) + \lim_{b\to\infty}\frac{1}{4}\tan^{-1}(b/4) = \frac{\pi}{8}+\frac{\pi}{8} = \frac{\pi}{4}.$$

   The integral *converges.*

4. $\displaystyle\int_1^2 \frac{1}{\sqrt[3]{x-1}}\,dx$

*Solution:* Improper because the function is discontinuous at $a = 1$ (an infinite discontinuity). Using a $u$-substitution, the integral is

$$\lim_{a\to 1^+} \frac{3}{2}(x-1)^{2/3}\Big|_a^2 = \lim_{a\to 1^+} \frac{3}{2}(1-(a-1)^{2/3}) = \frac{3}{2}.$$

Since the limit is finite, the integral converges. (Note this would not be true if the exponent of the $x - 1$ in the original integral was $\geq 1$.) The original

5. $\displaystyle\int_0^2 \frac{1}{x^2-5x+6}\,dx$

*Solution:* Improper because the function is discontinuous at $x = 2$. By partial fractions, the integral is $-\ln|x-2| + \ln|x-3|$. Then

$$\lim b \to 2^- - \ln|b-2| + \ln|b-3| + \ln|-2| - \ln|-3|$$

does not exist because $y = \ln|x|$ has a vertical asymptote at $x = 0$. This integral *diverges*.

6. $\displaystyle\int_1^3 \frac{1}{x^2-4x+4}\,dx$

*Solution:* This integral is improper because the function

$$\frac{1}{x^2-4x+4} = \frac{1}{(x-2)^2}$$

is discontinuous at $x = 2$ in the interior of the interval of integration (an infinite discontinuity). We need to see whether

$$\lim_{b\to 2^-} \int_1^b \frac{1}{(x-2)^2}\,dx$$

and

$$\lim_{a\to 2^+} \int_a^3 \frac{1}{(x-2)^2}\,dx$$

exist. But actually neither of them does because

$$\int \frac{1}{(x-2)^2}\,dx = -\frac{1}{x-2}$$

still has an infinite discontinuity at 2. This says that *neither limit above* exists. This integral *diverges*. (Note: The conclusion would be the same if either one of the limits failed to exist; it is not necessary that both fail to exist for the integral to diverge).