# College of the Holy Cross, Spring 2020
# Math 136, section 1, Solutions for Midterm Exam 1
# Friday, February 21

I.

A. (10) Evaluate the $L_4$ Riemann sum for $f(x) = e^{x^2}$ on the interval $[a, b] = [0, 2]$.

*Solution:* We have $\Delta x = \frac{2-0}{4} = \frac{1}{2}$, so the partition is $0 < \frac{1}{2} < 1 < \frac{3}{2} < 2$. The $L_4$ Riemann sum is

$$\begin{aligned} L_4 &= \sum_{j=1}^{4} e^{x_{j-1}^2} \Delta x \\ &= e^0 \cdot \frac{1}{2} + e^{1/4} \cdot \frac{1}{2} + e^1 \cdot \frac{1}{2} + e^{9/4} \cdot \frac{1}{2} \\ &\doteq 7.245 \end{aligned}$$

B. (10) The following limit of a sum would equal the definite integral $\int_a^b f(x)\,dx$ for some function $f(x)$ on some interval $[a, b]$. What function and what interval?

$$\lim_{N \to \infty} \sum_{j=1}^{N} \sqrt{\left(\frac{5(j-1)}{N}\right)^3 + 4} \cdot \frac{5}{N}.$$

*Solution:* The integral is $\displaystyle\int_0^5 \sqrt{x^3 + 4}\; dx$. (The sum is the $L_N$ Riemann sum for $f(x) = \sqrt{x^3 + 4}$ on $[0, 5]$ and the limit of the Riemann sums gives the integral.)

II. All parts of this problem refer to

$$A(x) = \int_0^x t(5 - t)\; dt$$

A. (10) Is $A(4)$ a positive or negative number? How can you tell?

*Solution:* $A(4)$ is positive since $f(x) = x(5-x) = 5x - x^2$ is $\geq 0$ for all $x$ in the interval $[0, 4]$–the graph $y = f(x)$ is a parabola opening down and crossing the $x$-axis at $x = 0$ and $x = 5$.

B. (10) Where does $A(x)$ have critical points? Is each of them a local maximum, an local minimum, or neither?

*Solution:* $A'(x) = x(5 - x)$ by the FTC, part II. This equals 0 at $x = 0$ and $x = 5$. Those are the critical points. Since $f(x) = A'(x)$ changes from negative to positive at 0, $A(x)$ has a local minimum there. Since $f(x) = A'(x)$ changes from positive to negative at 5, $A(x)$ has a local maximum there.

C. (10) If $B(x) = A(\tan(x)) = \int_0^{\tan(x)} t(5-t)\ dt$, find $B'(x)$.

*Solution:* By the FTC part II and the chain rule:

$$B'(x) = A'(\tan(x))\sec^2(x) = \tan(x)(5-\tan(x))\sec^2(x).$$

III.

A. (10) Integrate with a suitable $u$-substitution: $\displaystyle\int (6x^5+1)^{2/5}x^4\ dx$.

*Solution:* Let $u = 6x^5+1$. Then $du = 30x^4\ dx$. So $x^4\ dx = \frac{1}{30}\ du$ and the integral becomes

$$\int u^{2/5}\frac{1}{30}\ du = \frac{1}{30}\cdot\frac{5}{7}u^{7/5} + C = \frac{1}{42}(6x^5+1)^{7/5} + C.$$

B. (10) Integrate by parts: $\displaystyle\int x^2\cos(4x)\ dx$

*Solution:* We integrate by parts twice, letting $u = x^2$, then $u = x$:

$$\begin{aligned}
\int x^2\cos(4x)\ dx &= \frac{x^2\sin(4x)}{4} - \frac{2}{4}\int x\sin(4x)\ dx \\
&= \frac{x^2\sin(4x)}{4} - \frac{1}{2}\left(\frac{-x\cos(4x)}{4} + \frac{1}{4}\int\cos(4x)\ dx\right) \\
&= \frac{x^2\sin(4x)}{4} + \frac{x\cos(4x)}{8} - \frac{\sin(4x)}{32} + C
\end{aligned}$$

C. (10) Integrate: $\displaystyle\int \sec^5(x)dx$

*Solution:* Using the ST3 (twice) and ST4 formulas:

$$\int \sec^5(x)\ dx = \frac{\sec^3(x)\tan(x)}{4} + \frac{3\sec(x)\tan(x)}{8} + \frac{3}{8}\ln|\sec(x)+\tan(x)| + C$$

IV. (20) Integrate $\displaystyle\int_{-6}^{6} x^2\sqrt{36-x^2}\ dx$

*Solution:* From the form of the integrand, we know we want to use the trigonometric substitution $x = 6\sin\theta$, with $dx = 6\cos\theta\ d\theta$. Converting everything, including the limits of integration, we have

$$1296\int_{-\pi/2}^{\pi/2}\sin^2\theta\cos^2\theta\ d\theta$$

For this integral we can use either the SC 3 or SC 4 reduction formula. Here's how it turns out if you use SC 3 to reduce the power of sine, then SC 2 to reduce the power of cosine:

$$
\begin{aligned}
&= 1296\left(\frac{-\sin\theta\cos^3\theta}{4}\bigg|_{-\pi/2}^{\pi/2} + \frac{1}{4}\int_{-\pi/2}^{\pi/2}\cos^2\theta\ d\theta\right)\\
&= 1296\left(0 + \frac{\cos\theta\sin\theta}{8}\bigg|_{-\pi/2}^{\pi/2} + \frac{\theta}{8}\bigg|_{-\pi/2}^{\pi/2}\right)\\
&= 1296\left(0 + 0 + \frac{\pi}{8}\right)\\
&= 162\pi.
\end{aligned}
$$

Note: You can also convert the indefinite integral back to an equivalent function of $x$ and subsitute the original limits of integration $x = -6$ and $x = 6$.