MATH 135 – Calculus 1
The Limit Laws
September 20, 2019

Background

The Limit Laws from section 2.3 and today's video give a number of results that say we can evaluate limits by breaking down the function involved into simpler pieces. Provided each piece has a limit separately, we can combine those limits to get the limit of a more complicated function.

Questions

(1) Without looking at our book (or any notes you may have taken from today's video), write out what the Limit Sum, Product, and Quotient Laws tell us in your own words. (Then check your work by looking at the big boxed display of Theorem 1 on page 72 of our text.) *Comment:* Especially if you go on to more advanced mathematics courses in college, one of the things you will be asked to do more and more is to learn the statements of "big results" like this one. (Eventually you will need to learn the proofs too!)

(2) After completing question (1) and reviewing what those Limit Laws say, for each of the following limits *either* use the Laws to evaluate the limit (justifying every step by quoting the Law that applies), or else say why the Laws do not apply to that case:

(a) $\lim_{x\to 2} x^3 - 4x^2 + 2x + 7 = \lim_{x\to 2} x^3 - 4\cdot\lim_{x\to 2} x^2 + 2\lim_{x\to 2} x + 7$ (i) and (ii)
$= 8 - 16 + 4 + 7 = \boxed{3}$ (iii) or (v)

(b) $\lim_{x\to 0} \dfrac{3^x - 1}{x}$ ~ this is a $\frac{0}{0}$ form. The basic laws $\boxed{\text{don't apply}}$

(c) $\lim_{x\to 5} \dfrac{x^2+1}{x^2+2x+4} \overset{(iv)}{=} \dfrac{\lim_{x\to 5} x^2+1}{\lim_{x\to 5} x^2+2x+4} \overset{(i)+(ii)}{=} \dfrac{\lim_{x\to 5} x^2 + 1}{\lim_{x\to 5} x^2 + 2\lim_{x\to 5} x + 4} = \dfrac{26}{39} = \boxed{\dfrac{2}{3}}$

(d) $\lim_{t\to 4} \dfrac{\sqrt{t^2+9}}{t^{3/2}+6} \overset{(iv)}{=} \dfrac{\lim_{t\to 4}\sqrt{t^2+9}}{\lim_{t\to 4} t^{3/2}+6} \overset{(v)+(i)}{=} \dfrac{\sqrt{\lim_{t\to 4} t^2+9}}{\lim_{t\to 4} t^{3/2}+6} = \boxed{\dfrac{5}{14}}$

(e) $\lim_{z\to 3}(z^3 - 20)^2 \overset{(v)}{=} \left(\lim_{z\to 3} z^3 - 20\right)^2 \overset{2(ii)\text{ or }(v)}{=} 7^2 = \boxed{49}$

(3) Is it possible for $\lim_{x\to 0}(f(x) + g(x))$ to exist while $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist separately? If so, give an example where that happens. If not, say why not.

Yes: $f(x) = 1 - \frac{1}{x}$ $g(x) = 1 + \frac{1}{x}$ is an example $\lim_{x\to 0}\frac{1}{x}$ DNE, but the $\frac{1}{x}$ terms cancel in $f(x)+g(x)$

(4) It definitely is possible for $\lim_{x\to 0}(f(x)\cdot g(x))$ to exist while $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist separately. Find an example where that happens. (Hint: The easiest examples come by looking at piece-wise defined functions.)

Something like $f(x) = g(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ -1 & \text{for } x < 0 \end{cases}$ is an example

$\lim_{x\to 0} f(x)$ DNE since $\lim_{x\to 0^+} f(x) = 1 \neq -1 = \lim_{x\to 0^-} f(x)$ and same for $g(x)$

But $f(x)\cdot g(x) = 1$ for all x, so $\lim_{x\to 0} f(x)\cdot g(x) = 1$.