College of the Holy Cross
MATH 135, section 1
Solutions for Midterm Exam 3 – Friday, December 6

I. For each of the following functions find the derivative. You do not need to simplify.

A. (5) $f(x) = e^x \sin(5x) + \sec(x)$

Solution: By the product and chain rules, together with the derivative rules for $e^u, \sin(u), \sec(u)$, the derivative is

$$f'(x) = 5e^x \cos(5x) + e^x \sin(5x) + \sec(x)\tan(x).$$

B. (5) $g(x) = \dfrac{\cos(x)}{\ln(x)}$

Solution: By the quotient rule and the derivative rules for $\cos(u)$ and $\ln(u)$,

$$g'(x) = \frac{-\ln(x)\sin(x) - \cos(x)\cdot\frac{1}{x}}{(\ln(x))^2}.$$

C. (5) $h(x) = \tan^{-1}(x^2 - 4)$

Solution: By the Chain Rule and the derivative rule for the inverse tangent, the derivative is

$$h'(x) = \frac{2x}{1 + (x^2 - 4)^2}.$$

D. (5) Find $\frac{dy}{dx}$ by implicit differentiation if $x^2y^3 + 2y = 3$.

Solution: We have

$$x^2 \cdot 3y^2 \frac{dy}{dx} + 2xy^3 + 2\frac{dy}{dx} = 0,$$

so solving for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = \frac{-2xy^3}{3x^2y^2 + 2}.$$

E. (5) Find all the c in the open interval $(0, 4)$ for which the conclusion of the Mean Value Theorem is true for $f(x) = x^3 - 4x^2 + 3x$ on the interval $[a, b] = [0, 4]$

Solution: We are looking for c where $f'(c) = \frac{f(4)-f(0)}{4-0} = \frac{12}{4} = 3$. We have $f'(x) = 3x^2 - 8x + 3$, so the equation $f'(x) = 3$ is equivalent to $3x^2 - 8x = 0$. The roots are $x = 0, 8/3$. Of these, only $c = 8/3$ is in the open interval $(0, 4)$.

II. A poster is to be printed on a rectangular sheet of paper with 1 inch margins along the sides and 1.5 inch margins at the top and bottom. The rectangular printed region inside the margins must be 500 square inches in area. Find the dimensions of the overall sheet of the smallest possible area meeting these requirements. Let x, y be the width and height *of the printed area.*

A. (5) Draw a picture representing the whole poster and label the sides of length x and y clearly.

Solution: The inner rectangle (the printed area) should have width x and height y; the outer rectangle has width $x+2$ and height $y+3$.

B. (10) Express the area of the whole poster in terms of the one variable x–the width of the printed area.

Solution: Since $xy = 500$, $y = \frac{500}{x}$. Then the area of the whole poster is:

$$A = (x+2)(y+3) = (x+2)\left(\frac{500}{x} + 3\right) = 506 + \frac{1000}{x} + 3x$$

C. (10) Determine a critical point of your function from B. Why is it a minimum?

Solution: The derivative is

$$A'(x) = \frac{-1000}{x^2} + 3$$

This is defined for all $x > 0$. There is one positive critical point $x > 0$:

$$\frac{1000}{x^2} = 3 \Leftrightarrow x = \sqrt{1000/3} \doteq 18.26 \text{ inches}$$

This is a minimum of the area function by the Second Derivative Test since

$$A''(x) = \frac{2000}{x^3} > 0$$

for all positive x, showing that the graph is concave up for all $x > 0$. (Note: This can also be done by the First Derivative Test.)

D. (5) What are the dimensions of the whole poster of minimum area?

Solution: The width is $18.26 + 2 \doteq 20.26$ inches. The height is $\frac{500}{18.26} + 3 \doteq 30.38$ inches.

III. All parts of this question refer to the plot in Figure 1, which shows $y = f'(x)$ for some function f.

A. (10) Recall that the plot shows $y = f'(x)$(!) Approximately where does the graph $y = f(x)$ have inflection points?

Solution: Inflection points at $x \doteq 0.0, 0.6, 1.7$ (where $f'(x)$ changes from increasing to decreasing or vice versa).

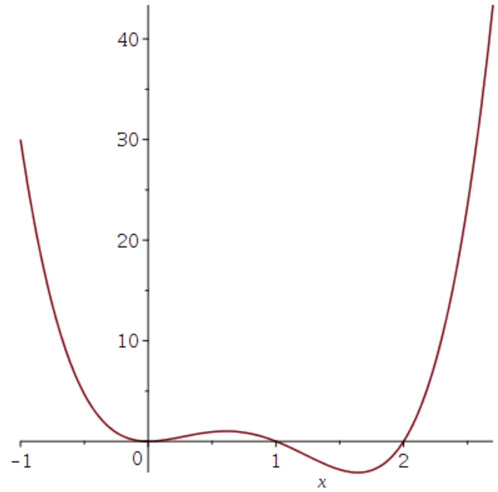

Figure 1: Figure for problem III

B. (10) Recall that the plot shows $y = f'(x)$(!) Where are the critical points of f and what types are they (i.e. local maximum, local minimum, neither)?

Solution: The critical points are $x = 0, 1, 2$. By the First Derivative Test, f has a local maximum at $x = 1$ (f' changes from positive to negative there), and a local minimum at $x = 2$ (f' changes sign from negative to positive there). The critical point at $x = 0$ is neither a maximum nor a minimum since the sign of $f'(x)$ does not change there.

IV. All parts of this question refer to the function $f(x) = \dfrac{x}{(2x+1)^2}$, for which the first two derivatives are, in simplified form:

$$\begin{aligned} f'(x) &= \frac{1-2x}{(2x+1)^3} \\ f''(x) &= \frac{8x-8}{(2x+1)^4} \end{aligned}$$

A. (10) Does $y = f(x)$ have any horizontal asymptotes? Use L'Hopital's Rule.

Solution: Yes, at $y = 0$. Since

$$\lim_{x\to\infty} \frac{x}{(2x+1)^2}$$

is an ∞/∞ indeterminate form limit, we can apply L'Hopital and this equals

$$\lim_{x\to\infty} \frac{1}{8x+4} = 0.$$

B. (10) What is the absolute maximum value of $f(x)$ on the interval $[0, 3]$?

Solution: $f'(x) = 0$ only at $x = 1/2$, which is in the interval. Note: The vertical asymptote at $x = -1/2$ is not in this interval, so f is continuous on $[0, 3]$. We have $f(0) = 0$, $f(1/2) = 1/8 = .125$ and $f(3) = \frac{3}{49} \doteq .061$. The minimum value is $f(0) = 0$ and the maximum value is $f(1/2) = 1/8$

C. (5) Over which interval(s) is the graph $y = f(x)$ concave up?

Solution: $f''(x) > 0$ for all $x > 1$ (we just need $8x - 8 > 0$, since the denominator is always positive). Concave up on: $(1, \infty)$.