MATH 135 – Calculus 1
Limits at Infinity and Horizontal Asymptotes
October 4, 2019

Background

Before the development of graphing calculators (i.e. as recently as about 20 years ago!) mathematics students generally learned to draw reasonably accurate sketches of fairly complicated graphs by hand and used that skill any time they wanted to understand the shape and properties of a function. I, for one, believe that something important *has been lost* now that technology has made it possible to do those routine tasks in an automated way and students are not being taught these skills any more. Namely, people learning mathematics today are not developing the sort of real "hands-on" familiarity with graphs that they used to get through the process of drawing many graphs by hand. Does this mean I believe that all graphs should be drawn by hand? *Of course not.* In fact, I regularly use mathematical software called *Maple* to draw the graphs in the handouts for this course (which are superior to what you could get from any calculator). But the point is that I *could draw* those graphs by hand if I had to, and I know enough about what the graph should look like that I can spot cases where I have made a typing mistake and entered a different function from the one I wanted, or where some aspect of the way the software works is producing an inaccurate picture!

In today's video, we saw what it means to say $\lim_{x\to\pm\infty} f(x) = L$ and we saw the connection with *horizontal asymptotes* to the graph $y = f(x)$. The goal of today's questions is to use all of that, together with other algebraic techniques to generate a good plot of the graph

$$y = \frac{(x-1)(x-3)}{(x-4)^2}$$

"the old way" – without using a graphing calculator.

Questions

(1) First, where does this function have discontinuities? Are there any vertical asymptotes? Where are they? What does the function do on either side of each of the discontinuities?
Infinite discontinuity at $x=4$. $\lim_{x\to 4^-} \frac{(x-1)(x-3)}{(x-4)^2} = +\infty$, $\lim_{x\to 4^+} \frac{(x-1)(x-3)}{(x-4)^2} = +\infty$

(2) Does this graph have a horizontal asymptote? Where is it? Does it cross that horizontal asymptote anywhere? Yes, $\lim_{x\to\pm\infty} \frac{(x-1)(x-3)}{(x-4)^2} = \lim_{x\to\pm\infty} \frac{x^2-4x+3}{x^2-8x+16} = \lim_{x\to\pm\infty} \frac{1-\frac{4}{x}+\frac{3}{x^2}}{1-\frac{8}{x}+\frac{16}{x^2}} = 1$ $\boxed{y=1}$

(3) Where does the graph cross the x-axis? Where does it cross the y-axis?
x-axis at $\boxed{x=1,3}$ y-axis at $\boxed{y=3/16}$

(4) How many points are there on each horizontal line $y = c$? (Hint: this means, how many different solutions are there of the equation: $\frac{(x-1)(x-3)}{(x-4)^2} = c$ for any given constant c? That equation can also be rewritten as $(x-1)(x-3) = c(x-4)^2$.) See back

(5) There is exactly one c for which the equation above has a repeated root (a double root). What is it? What happens if c is less than that value? See back.

(6) If you put together all of the information you have generated in the previous parts you should be able to get a reasonable sketch of the graph $y = f(x)$. Do it and then try to identify which of the three graphs on the back of this sheet is this function. Plot $\boxed{C}$

(4+5) $(x-1)(x-3) = c(x-4)^2$

when

$$x^2 - 4x + 3 = cx^2 - 8cx + 16c$$

or $\boxed{(c-1)x^2 + (4-8c)x + 16c - 3 = 0}$ (*)

By the quadratic formula,

$$x = \frac{(8c-4) \pm \sqrt{(4-8c)^2 - 4(c-1)(16c-3)}}{2(c-1)}$$

$$= \frac{8c-4 \pm \sqrt{12c+4}}{2(c-1)}$$

there are two real roots if

$12c+4 > 0$ and $c \neq 1$

so $c > -\frac{1}{3}$ and $c \neq 1$

there are no real roots when $c < -\frac{1}{3}$. there is a real double root when $c = -\frac{1}{3}$. The value $c = 1$ is a special case since equation (*) above is not quadratic

$$-4x + 13 = 0$$

$$\boxed{x = 13/4}$$

this is a point where the graph crosses the horizontal asymptote $\boxed{y=1}$.

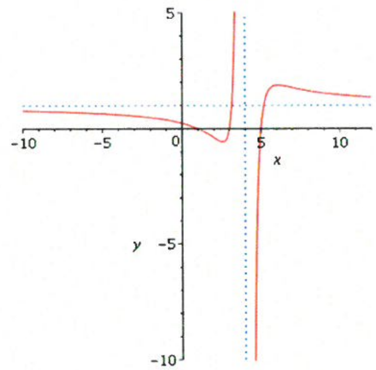

Figure 1: Plot A

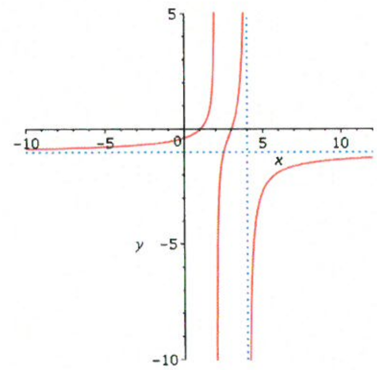

Figure 2: Plot B

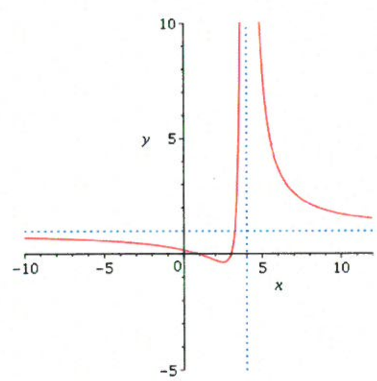

Figure 3: Plot C