MATH 135 – Calculus 1
Higher Derivatives
October 25, 2019

Background

If $f(x)$ is a function, $f'(x)$ is often called its *first derivative.* (In the alternate notation that might be written $\frac{dy}{dx}$ if we're thinking of the graph $y = f(x)$. The reason for this is that it is possible to go on and differentiate $f'(x)$ to get another new function. The derivative of $f'(x)$, that is, $(f')'(x)$ is also called the *second derivative* of the original f, and written $f''(x)$ or $\frac{d^2y}{dx^2}$. Continuing in the same way, if we can differentiate $f''(x)$, the result is called the *third derivative* of f, and so forth. The rules for computing these *higher derivatives* are exactly the same as the rules for computing $f'(x)$ to start. Today, we want to practice with these and understand why they are interesting.

Questions

(1) For each function, use the appropriate short-cut rules to find the first derivative, and then differentiate again to get the second derivative:

(a) $f(x) = x^5 + 4x^3 + x$. Also find the third derivative $f'''(x)$, the fourth derivative, the fifth derivative, and the sixth derivative for this one. (What *always* happens if you differentiate a polynomial function repeatedly enough times?)

Answer:

$$\begin{aligned}
f'(x) &= 5x^4 + 12x^2 + 1 \\
f''(x) &= 20x^3 + 24x \\
f'''(x) &= 60x^2 + 24 \\
f''''(x) &= 120x \\
f'''''(x) &= 120 \\
f''''''(x) &= 0
\end{aligned}$$

If you start from a polynomial and differentiate repeatedly, the derivative will eventually become *zero*, since the degree goes down by one each time. *Comment:* You can probably also see from the above that adding more and more primes is not a good notation because it becomes difficult to tell how many of them there are. Most mathematicians write $f^{(4)}(x)$ for the 4th derivative, and so forth. The parentheses tell you that this a higher derivative, not an exponent.

(b) $g(x) = \dfrac{x}{x^2 - 1}$. Your life will be a lot easier here if you simplify the first derivative before differentiating again to get $g''(x)$.

Answer: Using the quotient rule and simplifying,

$$\begin{aligned} g'(x) &= \frac{(x^2-1)\cdot 1 - x\cdot 2x}{(x^2-1)^2} \\ &= \frac{-x^2-1}{x^4-2x^2+1}. \\ \text{Then} g''(x) &= \frac{(x^4-2x^2+1)(-2x)+(x^2+1)(4x^3-4x)}{(x^4-2x^2+1)^2} \\ &= \frac{2x^5+4x^3-6x}{(x^4-2x^2+1)^2} \end{aligned}$$

(c) $h(x) = (x^2+x+1)e^x$. Also find the third derivative $h'''(x)$ for this one.

Answer:

$$\begin{aligned} h'(x) &= (x^2+3x+2)e^x \\ h''(x) &= (x^2+5x+5)e^x \\ h'''(x) &= (x^2+7x+10)e^x \end{aligned}$$

(2) So *why* would we want to be able to differentiate multiple times? The answer is that the second derivative f'' in particular encodes interesting information about the original function f.

(a) (A physical reason) – If $x(t)$ is the position of a moving object, then the rate of change of position $v(t) = x'(t)$ is called the *(instantaneous) velocity* at t. The rate of change of velocity is $v'(t) = x''(t)$. What is the physical name for the rate of change of velocity?

Answer: The *acceleration.*

(b) Suppose we know $f''(x) > 0$ on some interval (a, b). Recall that $f'' = (f')'$. What can we say about f' on that interval? Draw pictures illustrating graphs on which $f''(x) > 0$ for all x. What is the name for the property you are seeing (recall today's video)?

(c) Now, suppose we know $f''(x) < 0$ on some interval (a, b). Recall again that $f'' = (f')'$. What can we say about f' on that interval? Draw pictures illustrating graphs on which $f''(x) < 0$ for all x. What is the name for the property you are seeing?

Answer for b and c: On an interval where $f''(x) > 0$, $f'(x)$ is increasing. This means that the slopes of the tangent lines to $y = f(x)$ are increasing. On an interval where $f''(x) < 0$, $f'(x)$ is decreasing. This means that the slopes of the tangent lines to $y = f(x)$ are decreasing. In the first case, we say the graph is *concave up*; in the second, we say the graph is *concave down.*

(3) One of the graphs on the back this sheet is $y = f(x)$, and the other two are $y = f'(x)$ and $y = f''(x)$ for the same function $f(x)$. Which graph is which? (Be careful – these are not polynomial functions, so counting x-axis intercepts or "turning points" might not give the correct result!) *Answer:* C is the derivative of A and A is the derivative of B. In other words, B is $y = f(x)$, A is $y = f'(x)$ and C is $y = f''(x)$. You can see this most clearly by matching horizontal tangents to one graph with zeroes (places where the graph crosses the x-axis) of another.

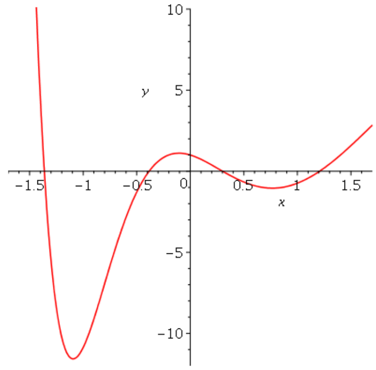

Figure 1: Plot A

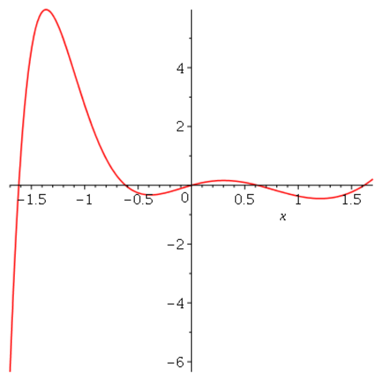

Figure 2: Plot B

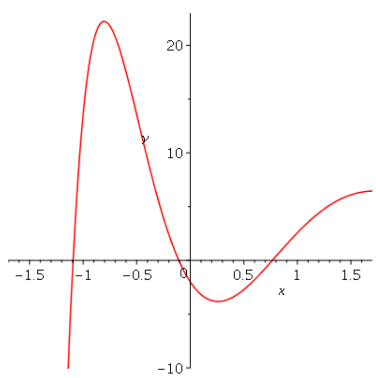

Figure 3: Plot C