MATH 135 – Calculus 1

The Squeeze Theorem and Trigonometric Limits

October 2, 2019

Background

In today's video, we saw an additional technique for evaluating limits called the "Squeeze Theorem" and the very important limit:

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1. \tag{1}$$

Recall that the Squeeze Theorem says: Assume $l(x) \le f(x) \le u(x)$ on some interval containing $x = c$ (except possibly at $x = c$) and $\lim_{x\to c} l(x) = \lim_{x\to c} u(x) = L$ (we might say "f is squeezed by l and u at $x = c$" to describe this). Then $\lim_{x\to c} f(x)$ exists and equals L as well.

Questions

Do the following problems from Section 2.6 in our text:

(1) Suppose the graphs $y = f(x), y = l(x), y = u(x)$ are as in the plot on the back. What can we say about $\lim_{x\to 1} f(x)$? $\lim_{x\to 1} f(x) = 2$

(2) Determine $\lim_{x\to 0} f(x)$ given that $\cos(x) \le f(x) \le 1$ for all x. $\lim_{x\to 0} f(x) = 1$

(3) State whether the given inequality provides sufficient information to determine $\lim_{x\to 1} f(x)$, and if so, find the limit. (Hint: Draw pictures!)

(a) $4x - 5 \le f(x) \le x^2$ not enough to determine $\lim_{x\to 1} f(x)$ (not "squeezed")

(b) $2x - 1 \le f(x) \le x^2$ $\lim_{x\to 1} f(x) = 1$

(c) $4x - x^2 \le f(x) \le x^2 + 2$ $\lim_{x\to 1} f(x) = 3$

Evaluate the following limits using (1):

(4)
$$\lim_{t\to 0} \frac{\sin(t)}{8t}. = \lim_{t\to 0} \frac{1}{8} \cdot \frac{\sin(t)}{t} = \frac{1}{8}\cdot 1 = \boxed{\frac{1}{8}}$$

(5)
$$\lim_{t\to 0} \frac{\sin(8t)}{t}. = \lim_{t\to 0} \frac{\sin(8t)}{8t} \cdot 8 = 1 \cdot 8 = \boxed{8}$$

(Hint: For this one let $u = 8t$ and convert the t in the denominator to an equivalent expression in terms of u. Note that $t \to 0$ implies $u = 8t \to 0$ also.)

(6)
$$\lim_{t\to 0} \frac{\sin(3t)}{\sin(5t)}. = \lim_{t\to 0} \frac{\frac{\sin(3t)}{3t}}{\frac{\sin(5t)}{(5t)}} \cdot \frac{3}{5} = \boxed{\frac{3}{5}}$$

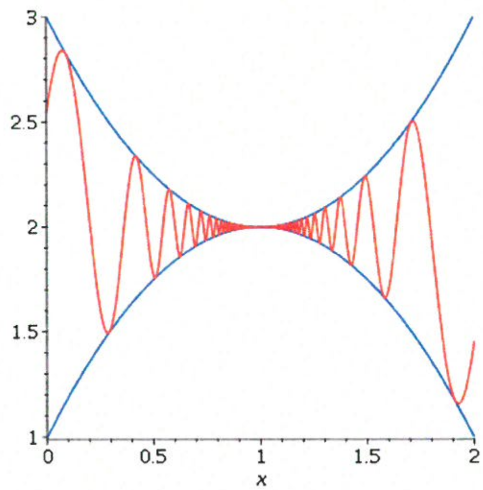

Figure 1: Plot for Question 1, $y = l(x), u(x)$ in blue; $y = f(x)$ in red.

(Hint: Rewrite as follows:

$$\frac{\sin(3t)}{\sin(5t)} = \frac{\frac{\sin(3t)}{t}}{\frac{\sin(5t)}{t}},$$

then proceed as in question (5).

(7)

$$\lim_{t\to 0} \frac{1-\cos(t)}{t^2} = \lim_{t\to 0} \frac{(1-\cos t)(1+\cos t)}{t^2(1+\cos t)}$$

$$= \lim_{t\to 0} \frac{\sin^2 t}{t^2} \cdot \frac{1}{1+\cos t}$$

$$= \lim_{t\to 0} \frac{\sin(t)}{t} \cdot \lim_{t\to 0} \frac{\sin(t)}{t} \cdot \lim_{t\to 0} \frac{1}{1+\cos t}$$

$$= 1 \cdot 1 \cdot \frac{1}{2}$$

$$= \boxed{\frac{1}{2}}$$