MATH 135 – Calculus 1
The Chain Rule and Implicit Differentiation
November 6, 2019

Background

As we saw in today's video, we may want to compute tangent lines to curves that are more general than graphs $y = f(x)$. (Equivalently we may want to understand rates of change of two related quantities when the relationship is more complicated than saying one is a function of the other.) For instance, consider the curve defined by the equation

$$x^3 + y^3 - 3xy = 0, \tag{1}$$

plotted on the back of this sheet. Note that this curve is *not a single graph* $y = f(x)$ because *it fails the vertical line test(!)*. However, by focusing on just portions of the curve, we can see that there are several graphs that lie on the curve. Each of those defines *y implicitly as a function of x* on a portion of the curve. These are called *implicit functions* because we will not have explicit formulas for them. However we can do things like compute derivatives because in (1), we can *think of y as a function of x* and differentiate things like y^3 by the chain rule, and things like the xy using the product rule. If we do this we get a new equation:

$$3x^2 + 3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} - 3y = 0.$$

Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}.$$

This process is called *implicit differentiation.* Then, for instance, if we wanted to find points where the tangent line was horizontal, we would set $\frac{dy}{dx} = 0$, and get $y = x^2$. So from (1),

$$x^3 + x^6 - 3x^3 = 0 \qquad \text{or} \qquad x^3(x^3 - 2) = 0.$$

This is satisfied when $x = 0$ and also $x = 2^{1/3} \doteq 1.26$. The point $(2^{1/3}, 2^{2/3})$ is one such point!

Questions

1. Find $\frac{dy}{dx}$ by implicit differentiation, then perform the other indicated calculations (if any):

   (a) $3x^2 + 2y^2 = 5$. Use $\frac{dy}{dx}$ to find the equation of the tangent line to this curve (an ellipse) at $(1, 1)$.

   Solution: Differentiating implicitly,

   $$6x + 4y\frac{dy}{dx} = 0$$

   so

   $$\frac{dy}{dx} = \frac{-6x}{4y} = \frac{-3x}{2y}.$$

   At $(x, y) = (1, 1)$, $\frac{dy}{dx} = \frac{-3}{2}$ so the tangent line is

   $$y - 1 = \frac{-3}{2}(x - 1), \quad \text{or} \quad y = \frac{-3x}{2} + \frac{5}{2}$$

(b) $\sin(xy) = x$.

Solution: Differentiating implicitly, using the product rule for the derivative of the "inside"

$$\cos(xy)\left(x\frac{dy}{dx} + y\right) = 1$$

so

$$\frac{dy}{dx} = \frac{1 - y\cos(xy)}{x\cos(xy)}.$$

(c) $xy + x^2y^2 = 6$. Find the equation of the tangent line to this curve at the point $(2, 1)$.

Solution: Differentiating implicitly, with the product rule on both terms:

$$x\frac{dy}{dx} + y + 2x^2y\frac{dy}{dx} + 2xy^2 = 0$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{-y - 2xy^2}{x + 2x^2y}.$$

At $(x, y) = (2, 1)$,

$$\frac{dy}{dx} = \frac{-1 - 4}{2 + 8} = \frac{-1}{2}$$

The tangent line is

$$y - 1 = \frac{-1}{2}(x - 2) \quad \text{or} \quad y = \frac{-x}{2} + 2$$

2. Find all the points on the curve defined by $y^2 = x^3 - 3x + 1$ where the tangent line is *horizontal*. (This curve is shown in Figure 10 on page 173 of our book.)

Solution: By implicit differentiation

$$\frac{dy}{dx} = \frac{3x^2 - 3}{2y}.$$

For a horizontal tangent line, we want $\frac{dy}{dx} = 0$ which happens at when $3x^2 - 3 = 3(x+1)(x-1) = 0$. However, note that with $x = 1$, we get $y^2 = -1$, which has no real solutions. With $x = -1$, we get $y^2 = 3$, so $y = \pm\sqrt{3}$. The two points where the tangent line is horizontal are $(-1, \sqrt{3}), (-1, -\sqrt{3})$.

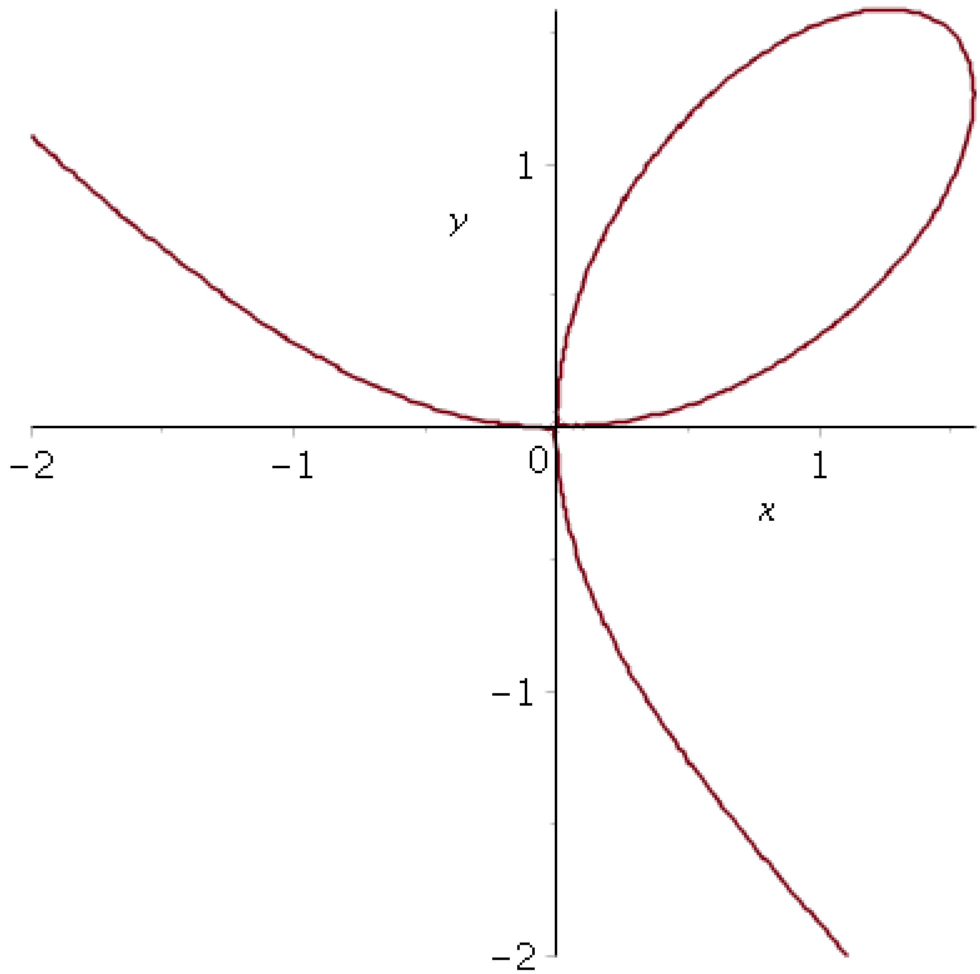

Figure 1: The curve $x^3 + y^3 - 3xy = 0$ (the "folium of Descartes")