MATH 135 – Calculus 1
Curve Sketching
November 20, 2019

Background

When mathematicians and scientists want to understand graphs of functions and other curves these days, they almost always turn to graphing calculators and mathematical software to generate the plots. So learning to plot functions by hand can seem somewhat pointless. However, in my experience, some experience with curve sketching by hand is a great way to solidify your understanding of the First and Second Derivative Tests, concavity, asymptotes, etc. So while you might never need to do this again in "real life," it's still a worthwhile educational exercise.

Here's a checklist of things to do and look for, given $y = f(x)$ to plot:

1. Determine the domain of your function and note whether the graph has any vertical asymptotes. Draw those in if there are any.
2. Compute $f'(x)$ and find the critical points $x = c$, and the critical values $f(c)$.
3. On each interval between adjacent critical points, determine the sign of $f'(x)$. Recall that intervals where $f'(x) > 0$ are intervals where f is increasing and intervals where $f'(x) < 0$ are intervals where f is decreasing.
4. Compute $f''(x)$ and determine any x where $f''(x) = 0$ or where $f''(x)$ does not exist.
5. Determine the intervals where f is concave up and concave down using the sign of f'' (recall that inflection points are places where the concavity changes. If your function has inflection points, find them and the function values there.
6. Determine horizontal asymptotes (if any), by taking $\lim_{x\to\pm\infty} f(x)$. (L'Hopital's Rule will be useful for some of these. One or other of these directions might not make sense for some functions if the domain does not extend to $\pm\infty$.)
7. The critical points and the inflection points can be thought of as *transition points* as described in Section 4.6 of the text.
8. You should now be prepared to sketch the graph using the transition points, the information about where the function is increasing, decreasing, concave up, and concave down. See Figures 1 and 2 on page 231 for the four basic local shapes of graphs and an example of how they can be put together.

Questions

Find the transition points (critical points and points of inflection), intervals of increase and decrease, intervals of upward and downward concavity, asymptotes, and sketch the graph:

1. $y = x^5 - 15x^3$ (Hint: This one has three critical points, three points of inflection, and no asymptotes.)

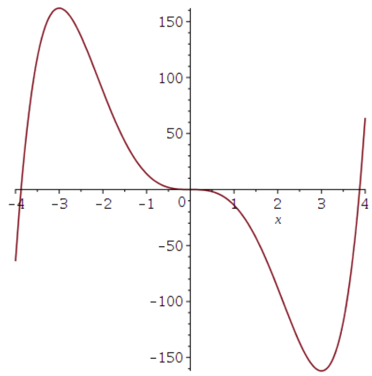

Figure 1: The graph $y = x^5 - 15x^3$ for $-4 \le x \le 4$

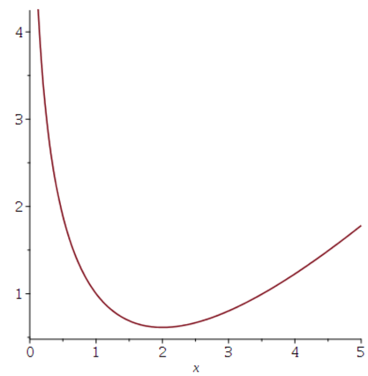

Figure 2: The graph $y = x - 2\ln(x)$ for $0 < x \le 5$

Answers: $y' = 5x^4 - 45x^2 = 5x^2(x^2 - 9)$. This exists for all x; it is zero at $x = -3, 0, 3$. y is increasing for $x < -3$, decreasing for $-3 < x < 3$, increasing for $x > 3$. By the first derivative test, y has a local maximum at $(x, y) = (-3, 162)$; the critical point at $x = 0$ is neither a local maximum nor local minimum, and y has a local minimum at $(x, y) = (3, -162)$. $y'' = 20x^3 - 90x = 10x(2x^2 - 9)$. This is zero at $x = -3/\sqrt{2}, 0, 3/\sqrt{2}$ The sign of y'' changes at each, so these are all points of inflection. The graph is concave down on $(-\infty, -3/\sqrt{2})$ and $(0, 3/\sqrt{2})$ and concave up on $(-3/\sqrt{2}, 0)$ and $(3/\sqrt{2}, \infty)$. The graph is given in Figure 1.

2. $y = x - 2\ln(x)$ (Hint: $\lim_{x\to 0+} \ln(x) = -\infty$.)

   Answers: The domain of this function is just the set of all $x > 0$. The graph has a vertical asymptote at $x = 0$ from the right, with $\lim_{x\to 0+} y = +\infty$. The first derivative is $y' = 1 - \frac{2}{x} = 0$ when $x = 2$. Note: We *do not* say that $x = 0$ is a critical point here since that x is not in the domain of y either. y is decreasing on $(0, 2)$ and increasing on $(2, \infty)$. So y has a local minimum at $(2, 2 - \ln(2))$. The second derivative is $y'' = \frac{2}{x^2}$. This is > 0 for all $x > 0$, so the graph is always concave up. It is given in Figure 2.

3. $y = \dfrac{1}{x^2 - 2x}$ (Hint: This one has both horizontal and vertical asymptotes. Be sure to simplify

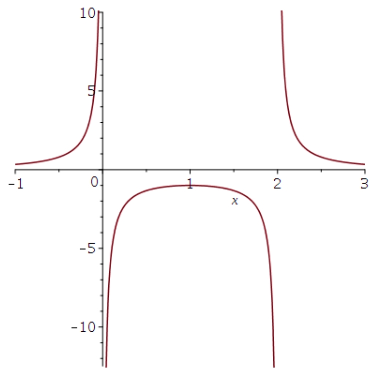

Figure 3: The graph $y = \frac{1}{x^2-2x}$ for $-1 \le x \le 3$

$f'(x)$ before computing the second derivative.)

4. *Answers*: There are vertical asymptotes at the roots of the polynomial in the denominator: $x = 0$ and $x = 2$. $\lim_{x\to\pm\infty} \frac{1}{x^2-2x} = 0$, so there is a horizontal asymptote at $y = 0$ (in both directions). We have $y' = \frac{-(2x-2)}{(x^2-2x)^2}$. This is zero at $x = 1$. $y' > 0$ for all $x < 1$ except $x = 0$, and $y' < 0$ for all $x > 1$ except $x = 2$. Hence $(1, -1)$ is a local maximum. Next,

$$y'' = \frac{(x^2-2x)^2(-2) + 2(2x-2)(x^2-2x)(2x-2)}{(x^2-2x)^4} = \frac{6x^2-12x+8}{(x^2-2x)^3}.$$

The top here is always > 0, but the bottom is > 0 for $x < 0$ and $x > 2$, while the bottom is < 0 for $0 < x < 2$. This says that the graph is concave up for $x < 0$ and $x > 2$, but concave down for $0 < x < 2$. Note: We would *not* call $x = 0, 2$ points of inflection here because y is not continuous at those points because of the vertical asymptotes. The graph is given in Figure 3.

5. $y = x^2e^{-2x}$ (Hint: This has a horizontal asymptote in one direction but not the other.)

Answers: $y' = (-2x^2 + 2x)e^{-2x}$ so there are critical points at $x = 0$ and $x = 1$. $y' < 0$ for $x < 0$ and $x > 1$ while $y' > 0$ for $0 < x < 1$. By the First Derivative Test, y has a local minimum at $(0, 0)$ and a local maximum at $(1, e^{-2})$. Then $y'' = (4x^2 - 8x + 2)$ this is $= 0$ at $x = \frac{8\pm\sqrt{64-32}}{8} = 1 \pm \frac{\sqrt{2}}{2} \doteq .293, 1.707$. These are points of inflection because $y'' > 0$ (the graph is concave up) for $x < 1-\frac{\sqrt{2}}{2}$ and $x > 1+\frac{\sqrt{2}}{2}$ while $y'' < 0$ (the graph is concave down) for $1 - \frac{\sqrt{2}}{2} < x < 1 + \frac{\sqrt{2}}{2}$. Finally, by L'Hopital's Rule,

$$\lim_{x\to\infty} \frac{x^2}{e^{2x}} = \lim_{x\to\infty} \frac{2x}{2e^{2x}} = \lim_{x\to\infty} \frac{2}{4e^{2x}} = 0,$$

so the graph has a horizontal asymptote at $y = 0$ (in the positive x-direction only). The graph is given in Figure 4.

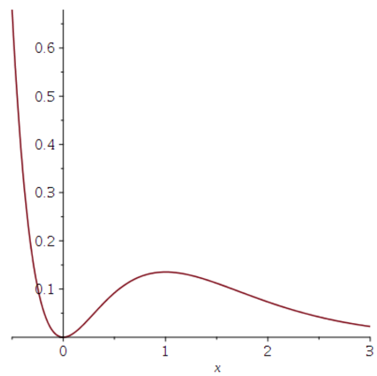

Figure 4: The graph $y = x^2e^{-2x}$ for $-1/2 \leq x \leq 3$